CONSTRUCTING DISTANCE MAGIC GRAPHS FROM REGULAR GRAPHS

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Abstract. A graph G with k vertices is *distance magic* if the vertices can be labeled with numbers $1, 2, \ldots, k$ so that the sum of labels of the neighbors of each vertex is equal to the same constant μ_0. We present a construction of distance magic graphs arising from arbitrary regular graphs based on an application of magic rectangles. We also solve a problem posed by Shafiq, Ali, and Simanjuntak.

1. Introduction

A graph G with the vertex set $V(G)$, edge set $E(G)$, and $|V(G)| = n$ is called *distance magic* if there exists an injective mapping

$$\mu : V \to \{1, 2, \ldots, n\}$$

such that the *weight* of each vertex x, defined as

$$w(x) = \sum_{xy \in E(G)} \mu(y),$$

is equal to the same constant μ_0, called the *magic constant*. The mapping is called a *distance magic labeling*. In some papers, μ is also called a 1-*vertex-magic vertex labeling* (e.g., [1,2]).

For a graph G, we denote by mG a graph consisting of m vertex-disjoint copies of G. In particular, tK_1 is then the complement of K_t and we will use the usual notation $\overline{K_t}$ instead. When H is an arbitrary graph

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with vertices $x_1, x_2, \ldots, x_n$, and G is any graph with t vertices, then by $H[G]$ we denote the graph, which arises from H by replacing each vertex x_i by a copy of the graph G with vertex set X_i, and each edge $x_i x_j$ by the edges of the complete bipartite graph $K_{t,t}$ with bipartition X_i, X_j. The graph $H[G]$ is then called the *lexicographic product* or the *composition* of H and G. In some literature this product is called the *wreath product* of H and G and denoted by $H \bullet G$.

It was proved by Miller, Rodger, and Simanjuntak that when H is an arbitrary regular graph (connected or disconnected), then $H[\overline{K_t}]$ is distance magic for any even t. (They stated their result in terms of the wreath product but denoted it by $H \times \overline{K_t}$.) They also proved that the complete regular p-partite graph $K_p[\overline{K_t}]$ has a distance magic labeling if and only if either t is even or both p and t are odd [6]. In a recent paper [7], Shafiq, Ali, and Simanjuntak found sufficient conditions for the existence of distance magic labelings of the graphs $mK_p[\overline{K_t}]$ and $mC_p[\overline{K_t}]$.

In this paper we extend their results by showing that when H is an arbitrary regular graph (connected or disconnected) with k vertices and k is odd, then $H[\overline{K_t}]$ is distance magic for any odd integer t. Our result completes the characterization of distance magic graphs $H[\overline{K_t}]$ when the the order of H is odd, because r-regular distance magic graphs for r odd do not exist, as shown by Miller, Rodger, and Simanjuntak in [6].

The methods used in the proofs of our result were already used by the authors in [1] and by the first author in [2]. The construction was mentioned (without proof) by Sugeng, Fronček, Miller, Ryan, and Walker in [8], where similar techniques were used in constructions of bi-regular distance magic graphs.

2. Known results

All feasible values of r of r-regular distance magic graphs with an even number of vertices were determined in [6] and [1].

Theorem A. [6] *If G is an r-regular distance magic graph, then r is even.*

Theorem B. [1] *For n even an r-regular distance magic graph with n vertices exists if and only if $2 \leq r \leq n-2, r \equiv 0 \pmod 2$ and either $n \equiv 0 \pmod 4$ or $r \equiv 0 \pmod 4$.*

For graphs with an odd number of vertices, the existence question of regular distance magic graphs was partially answered in [2].

Theorem C. [2] *Let n, q be odd integers and s an integer, $q \geq 3$, $s \geq 1$. Let $r = 2^s q$, $q \mid n$ and $n \geq r + q$. Then an r-regular distance magic graph of order n exists.*

When the maximum odd divisor of r does not divide n, somewhat weaker result can be proved.

Theorem D. [2] *Let n, q be odd integers and s an integer, $q \geq 3$, $s \geq 1$. Let $r = 2^s q$, $q \mid n$ and $n \geq \dfrac{7r+4}{2}$. Then an r-regular distance magic graph of order n exists.*

For even values of t, the question of the existence of a distance magic labeling of $H[\overline{K_t}]$ was settled in [6].

Theorem E. [6] *Let H be an arbitrary regular graph. Then the graph $H[\overline{K_t}]$ is distance magic for any positive even number t.*

Shafiq, Ali, and Simanjuntak found sufficient conditions for the existence of distance magic labelings of unions of complete bipartite regular graphs $mK_p[\overline{K_t}]$ and unions of "blown up cycles" $mC_p[\overline{K_t}]$.

Theorem F. [7] (i) *If t is even or mtp is odd, $m \geq 1, t > 1$ and $p > 1$, then $mK_p[\overline{K_t}]$ has a distance magic labeling.*
(ii) *If tp is odd, m is even, and $p \equiv 3 \pmod 4$, then $mK_p[\overline{K_t}]$ does not have a distance magic labeling.*

In the next section we show that Theorem F can be strengthened. For the graphs $mC_p[\overline{K_t}]$, Shafiq, Ali, and Simanjuntak found a complete characterization.

Theorem G. [7] *Let $m \geq 1$, $t > 1$ and $p \geq 3$. Then $mC_p[\overline{K_t}]$ has a distance magic labeling if and only if either t is even or mtp is odd or t is odd and $p \equiv 0 \pmod 4$.*

3. Construction

Now we present our construction, which extends the existence part of Theorem E to odd values of t. Our proof is based on an application of *magic rectangles.* They were first studied by T. Harmuth [4,5] in late 19th century. For a modern proof of the existence result, see T.R. Hagedorn [3].

Definition 1. A *magic rectangle* $\mathrm{MR}(a,b)$ is an $a \times b$ array with $a, b > 1$ in which the first ab positive integers are placed so that the sum over each column of $\mathrm{MR}(a,b)$ is $\sigma(a,b) = a(ab+1)/2$ and the sum over each row is $\tau(a,b) = b(ab+1)/2$.

Theorem H. [2,3,4] *A magic rectangle* $\mathrm{MR}(a,b)$ *exists if and only if $a \equiv b \pmod 2$ except when $a = b = 2$.*

Using Theorem H, we prove our main result.

Theorem 2. *Let H be an arbitrary r-regular graph with an odd number of vertices and t be an odd positive integer. Then r is even and the graph $H[\overline{K_t}]$ is distance magic.*

Proof. If r is odd, then we have a graph with an odd number of vertices of odd degree, which is absurd. Hence, r is even. We denote the vertices of H by $x_1, x_2, \ldots, x_k$. The vertex set of $H[\overline{K_t}]$ then consists of k sets $X_1, X_2, \ldots, X_k$, where $X_j = \{x_{j1}, x_{j2}, \ldots, x_{jt}\}$ for $j = 1, 2, \ldots, k$, and $x_{ic}x_{jd}$ is an edge of $H[\overline{K_t}]$ for any $c, d \in \{1, 2, \ldots, t\}$ if and only if x_ix_j is and edge of H. Label the vertices of each set X_j by the elements of the j-th column of $\mathrm{MR}(t, k)$, that is, set $\mu(x_{uv}) = f_{uv}$, where f_{uv} is the entry of $\mathrm{MR}(t, k)$ in row u and column v. The sum of the labels in each set X_j is $\sigma(t, k) = t(kt + 1)/2$. Because H is r-regular, the sum of labels of the neighbors of every vertex is $rt(kt + 1)/2$, which concludes the proof. □

On the other hand, we are able to prove the non-existence of distance magic labelings only for one special case of graphs $H[\overline{K_t}]$ not covered by Theorem 2.

Theorem 3. *Let t be odd, $k \equiv r \equiv 2 \pmod 4$, and H be an r-regular graph with k vertices. Then $H[\overline{K_t}]$ is not distance magic.*

Proof. Set $t = 2T + 1, k = 4K + 2, r = 4R + 2$. Then $H[\overline{K_t}]$ is rt-regular and

$$rt = (4R + 2)(2T + 1) = 8RT + 4T + 4R + 2$$

and $rt \equiv 2 \pmod 4$. At the same time, the number of vertices of $H[\overline{K_t}]$ is

$$n = kt = (4K + 2)(2T + 1) = 8KT + 4T + 4K + 2$$

and $n \equiv 2 \pmod 4$. But then by Theorem B, $H[\overline{K_t}]$ cannot be distance magic since no distance magic graph with these parameters exists. □

Finally, we strengthen Theorem F by solving the following problem posed by Shafiq, Ali, and Simanjuntak in [7].

Problem J. [7] For the graph $mK_p[\overline{K_t}]$, where m is even, t is odd, $p \equiv 1 \pmod 4$, and $p > 1$, determine if there is a distance magic labeling.

We solve the problem in the following theorem by showing that no such labeling is possible.

Theorem 4. *The graph $mK_p[\overline{K_t}]$, where m is even, t is odd, $p \equiv 1 \pmod 4$, and $p > 1$, is not distance magic.*

Proof. We proceed by contradiction and suppose to the contrary that the graph $mK_p[\overline{K_t}]$ is distance magic. Let $X_i = \{x_{ij} | j = 1, 2, \ldots, t\}$ for $i = 1, 2, \ldots, p$ be the partite sets in one of the components of $mK_p[\overline{K_t}]$. Then

$$w(x_{1a}) = \sum_{i=2}^{p} \sum_{j=1}^{t} \mu(x_{ij})$$

for every $a = 1, 2, \ldots, t$ and

$$w(x_{pb}) = \sum_{i=1}^{p-1} \sum_{j=1}^{t} \mu(x_{ij})$$

for every $b = 1, 2, \ldots, t$. Because $w(x_{1a}) = w(x_{pb}) = \mu_0$, it is obvious that

$$\sum_{j=1}^{t} \mu(x_{1j}) = \sum_{j=1}^{t} \mu(x_{pj})$$

and it can be easily observed that

$$\sum_{j=1}^{t} \mu(x_{1j}) = \sum_{j=1}^{t} \mu(x_{ij})$$

for every $i = 2, 3, \ldots, p$. Similar observation can be done for every component and therefore, the sum of weights of labels in every partite set in each component must be the same.

Because the sum of all labels in $mK_p[\overline{K_t}]$ is $mpt(mpt+1)/2$, we have

$$\sum_{j=1}^{t} \mu(x_{1j}) = \frac{mpt(mpt+1)}{2mp} = \frac{t(mpt+1)}{2}.$$

Since both t and $mpt+1$ are odd, the number $t(mpt+1)/2$ is not an integer and we get the desired contradiction. □

Theorem 4 along with part (ii) of Theorem F then give the following necessary and sufficient condition.

Corollary 5. *The graph $mK_p[\overline{K_t}]$, where tp is odd and m is even, $p > 1, m \geq 2$, is distance magic if and only if $p \equiv 3 \pmod 4$.*

Theorem 4 together with Theorem B also show that there is no nice necessary and sufficient condition in terms of parameters k, r, t for distance magic graphs $H[\overline{K_t}]$, when t is odd. That is so because for every combination of k and r even and t odd Theorem 4 shows an example of a graph $H[\overline{K_t}]$ which is *not* distance magic, while Theorem B guarantees (except when $k \equiv r \equiv 2 \pmod 4$) that for the same triple k, r, t that there *exist* distance magic graphs with these parameters.

This observation suggests that one might try to find a "nice" characterization of even-regular graphs H for which the products $H[\overline{K_t}]$ are distance magic.

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