

Putting Dots in Triangles

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Abstract

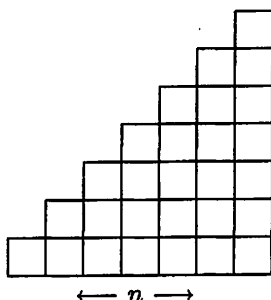
Given a right-angled triangle of squares in a grid whose horizontal and vertical sides are n squares long, let $N(n)$ denote the maximum number of dots that can be placed into the cells of the triangle such that each row, each column, and each diagonal parallel to the long side of the triangle contains at most one dot. It has been proven that $N_f(n) = \lfloor \frac{2n+1}{3} \rfloor$. In this note, we give a new proof of the upper bound $N_f(n) \leq \lfloor \frac{2n+1}{3} \rfloor$ using linear programming techniques.

1 The problem

Consider a ‘triangle’ of squares in a grid whose sides are n squares long, as illustrated by the diagram in Figure 1, for which $n = 7$. We call a southwest-to-northeast diagonal a *standard diagonal*. Note that our triangle consists of all the cells in an $n \times n$ square that lie on or below the longest standard diagonal.

We denote by $N(n)$ the maximum number of dots that can be placed into the cells of the triangle such that each row, each column, and each standard diagonal contains at most one dot. Determining $N(n)$ is equivalent to solving the following problem: Suppose we have a chessboard made up of hexagonal cells arranged in the shape of an equilateral triangle of side n . Then $N(n)$ is the maximum number of non-attacking queens that can

Figure 1: A triangle of size 7



be placed on such a board, where a queen can move in any one of the three directions allowed on a hexagonal grid. The following theorem was proven by Vaderlind, Guy and Larson [3, Problem 252] and independently by Nivasch and Lev [2]:

Theorem 1.1. $N(n) = N_f(n)$, where

$$\begin{aligned} N_f(3t) &= 2t \\ N_f(3t + 1) &= 2t + 1 \\ N_f(3t + 2) &= 2t + 1. \end{aligned}$$

Note that the value of $N_f(n)$ can be stated more succinctly as follows:

$$N_f(n) = \left\lfloor \frac{2n + 1}{3} \right\rfloor.$$

In order to prove Theorem 1.1, we require a construction to establish the lower bound $N(n) \geq N_f(n)$ as well as a proof of the upper bound $N(n) \leq N_f(n)$. In [2, 3], the upper bound was proven by elementary combinatorial arguments. In this note, we give a new proof of the upper bound using linear programming techniques. In the end, our proof is also combinatorial; the main contribution we make is to demonstrate the use of linear programming techniques in deriving the proof.

2 The lower bound

Before proving the upper bound, we give a construction to show that $N(n) \geq N_f(n)$. This construction is essentially the same as the ones in

[2, 3].

Theorem 2.1. $N(n) \geq N_f(n)$.

Proof. First, we show that $N(3t + 1) \geq 2t + 1$:

1. Place a dot in the leftmost cell of the $(2t + 1)$ st row (where we number rows from top to bottom).
2. Place t more dots, each two squares to the right and one square up from the previous dot.
3. Place a dot in the $(t + 2)$ nd cell from the left in the bottom row.
4. Place $t - 1$ more dots, each two squares to the right and one square up from the previous dot.

It is easily verified that at most one dot is contained in each row, column, or standard diagonal.

Next, $N(3t + 2) \geq N(3t + 1) \geq 2t + 1$ (it suffices to add a row of empty cells). Finally, $N(3t) \geq N(3t + 1) - 1 \geq 2t$ (delete the bottom row of cells from a triangle of side $3t + 1$, noting that any row contains at most one dot). \square

Example 2.1. We show in Figure 2 that $N(7) \geq 5$ by applying the construction given in Theorem 2.1.

3 A new proof of the upper bound

The computation of $N(n)$ can be formulated as an integer linear program. Suppose we number the cells as indicated in the Figure 3 (where $n = 6$):

Define $x_{i,j} = 1$ if the corresponding cell contains a dot; define $x_{i,j} = 0$ otherwise. The sum of the variables in each row, column, and standard diagonal is at most 1. This leads to constraints of the form

$$\sum_{j=1}^i x_{i,j} \leq 1, \quad \text{for } i = 1, 2, \dots, n$$

$$\sum_{i=j}^n x_{i,j} \leq 1, \quad \text{for } j = 1, 2, \dots, n$$

Figure 2: $N(7) \geq 5$

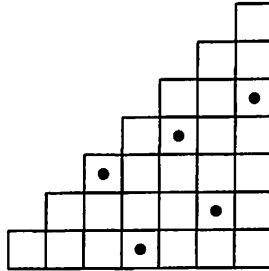
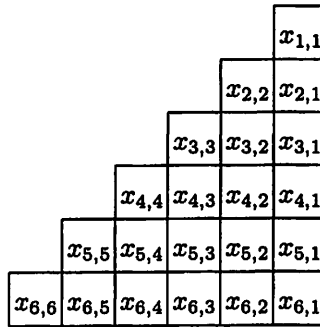


Figure 3: Labelling the cells in a triangle of size 6



and

$$\sum_{i=k+1}^n x_{i,i-k} \leq 1, \quad \text{for } k = 0, 1, \dots, n-1.$$

Finally, $x_{i,j} \in \{0, 1\}$ for all i, j . The objective function is to maximize $\sum x_{i,j}$ subject to the above constraints. It is obvious that the optimal solution to this integer program is $N(n)$.

It is possible to relax the integer program to obtain a linear program, replacing the condition $x_{i,j} \in \{0, 1\}$ by $0 \leq x_{i,j} \leq 1$ for all i, j . In fact, we do not have to specify $x_{i,j} \leq 1$ as an explicit constraint since it is already implied by the other constraints; it suffices to require $0 \leq x_{i,j}$ for all i, j . Denoting the optimal solution to this linear program by $LP(n)$, we have

Figure 4: The optimal solution to $LP(6)$

					0
				0	$\frac{2}{7}$
			0	$\frac{5}{7}$	$\frac{2}{7}$
		$\frac{5}{7}$	0	0	$\frac{2}{7}$
	$\frac{2}{7}$	0	$\frac{3}{7}$	$\frac{1}{7}$	$\frac{1}{7}$
0	0	$\frac{2}{7}$	$\frac{4}{7}$	$\frac{1}{7}$	0

that $LP(n) \geq N(n)$.

Example 3.1. Using Maple, it can be seen that $LP(6) = 4\frac{2}{7}$ (a solution to the LP having this value is presented in Figure 4).

Next, we tabulate some solutions to $LP(n)$ for small values of n in Table 1. Based on the numerical data in Table 1, it is natural to formulate a conjecture about $LP(n)$:

Conjecture 3.1 (LP Conjecture). *Define*

$$\begin{aligned}
 LP_f(3t) &= 2t + \frac{t}{3t+1} \\
 LP_f(3t+1) &= 2t+1 \\
 LP_f(3t+2) &= 2t+1 + \frac{2t+1}{3t+2}.
 \end{aligned}$$

Then we conjecture that $LP(n) = LP_f(n)$.

It is easy to show the following:

Theorem 3.2. *If the LP Conjecture is true, then $N(n) = N_f(n)$.*

Proof. First, the LP Conjecture asserts that

$$LP(n) = LP_f(n). \tag{1}$$

Table 1: Optimal solutions to the integer and linear programs for small n

n	$N(n)$	$LP(n)$	$LP(n) - N(n)$
3	2	$2\frac{1}{4}$	$\frac{1}{4}$
4	3	3	0
5	3	$3\frac{3}{5}$	$\frac{3}{5}$
6	4	$4\frac{2}{7}$	$\frac{2}{7}$
7	5	5	0
8	5	$5\frac{5}{8}$	$\frac{5}{8}$
9	6	$6\frac{3}{10}$	$\frac{3}{10}$
10	7	7	0
11	7	$7\frac{7}{11}$	$\frac{7}{11}$
12	8	$8\frac{4}{13}$	$\frac{4}{13}$

Because $N(n)$ is an integer and $N(n) \leq LP(n)$, we have that

$$N(n) \leq \lfloor LP(n) \rfloor. \quad (2)$$

Simple arithmetic establishes that

$$\lfloor LP_f(n) \rfloor = N_f(n). \quad (3)$$

Combining (1), (2) and (3), we have

$$N(n) \leq \lfloor LP(n) \rfloor = \lfloor LP_f(n) \rfloor = N_f(n).$$

We showed in Theorem 2.1 that $N(n) \geq N_f(n)$; hence $N(n) = N_f(n)$. \square

The optimal solution to the linear program for $n = 6$ that we presented in Figure 4 does not seem to have much apparent structure that could be the basis of a mathematical proof. Indeed, most of the small optimal solutions that we obtained are quite irregular, which suggests that proving the LP conjecture could be difficult. We circumvent this problem by instead studying the dual LP and appealing to weak duality.

An LP in *standard form* is specified as:

maximize $c^T x$ subject to $Ax \leq b, x \geq 0.$

This is often called the *primal LP*. Any vector x such that $Ax \leq b$, $x \geq 0$ is called a *feasible solution*. The *objective function* is the value to be maximized, namely, $c^T x$.

The corresponding *dual LP* is specified as:

minimize $b^T y$ subject to $A^T y \geq c, y \geq 0$.

Here, a feasible solution is any vector y such that $A^T y \geq c$, $y \geq 0$. The objective function is $b^T y$.

We will use the following classic theorem.

Theorem 3.3 (Weak Duality Theorem). *The objective function value of the dual LP at any feasible solution is always greater than or equal to the objective function value of the primal LP at any feasible solution.*

We now describe the dual LP for our problem. Suppose we label the rows of our triangle by r_1, r_2, \dots, r_n , such that r_i is the row containing i squares, and we label the columns and diagonals similarly. The following simple lemma is very useful.

Lemma 3.4. *If a cell is in row r_i , column c_j and diagonal d_k , then $i + j + k = 2n + 1$.*

In fact, it is not hard to see that there is a bijection from the set of $n(n + 1)/2$ cells to the set of triples

$$\mathcal{T} = \{(i, j, k) : i + j + k = 2n + 1, 1 \leq i, j, k \leq n\}.$$

In the dual LP, the variables are $r_1, r_2, \dots, r_n, c_1, c_2, \dots, c_n, d_1, d_2, \dots, d_n$. There is a constraint for each cell C . If C is in row r_i , column c_j and diagonal d_k , then the corresponding constraint is

$$r_i + c_j + d_k \geq 1.$$

The objective function is to minimize $\sum r_i + \sum c_j + \sum d_k$.

It turns out that there exist optimal solutions for the dual LP that have a very simple, regular structure. These were obtained by extrapolating solutions for small cases found by Maple.

When $n = 3t + 1$, define

$$r_i = c_i = \max \left\{ 0, \frac{i - t - 1}{3t + 1} \right\}, \quad d_i = \max \left\{ 0, \frac{i - t}{3t + 1} \right\}. \quad (4)$$

When $n = 3t + 2$, define

$$r_i = c_i = d_i = \max \left\{ 0, \frac{i-t-1}{3t+2} \right\}. \quad (5)$$

When $n = 3t$, define

$$r_i = c_i = d_i = \max \left\{ 0, \frac{i-t}{3t+1} \right\}. \quad (6)$$

Lemma 3.5. *The values r_i, c_i and d_i defined in (4), (5) and (6) are feasible for the dual LP, and the value of the objective function for the dual LP at these solutions is $LP_f(n)$.*

Proof. First we consider the case $n = 3t + 1$. Consider any cell C , and suppose C is in row r_i , column c_j and diagonal d_k . Then we have that

$$\begin{aligned} r_i + c_j + d_k &\geq \frac{i-t-1}{3t+1} + \frac{j-t-1}{3t+1} + \frac{k-t}{3t+1} \\ &= \frac{i+j+k-(3t+2)}{3t+1} \\ &= \frac{6t+3-(3t+2)}{3t+1} \quad (\text{applying Lemma 3.4}) \\ &= 1. \end{aligned}$$

Therefore all constraints are satisfied. The value of the objective function is

$$\begin{aligned} &\frac{1}{3t+1} \left(\sum_{i=t+1}^{3t+1} (i-t-1) + \sum_{i=t+1}^{3t+1} (i-t-1) + \sum_{i=t}^{3t+1} (i-t) \right) \\ &= \frac{1}{3t+1} \left(\frac{2t(2t+1)}{2} + \frac{2t(2t+1)}{2} + \frac{(2t+1)(2t+2)}{2} \right) \\ &= \frac{(2t+1)(3t+1)}{3t+1} \\ &= 2t+1 \\ &= LP_f(3t+1). \end{aligned}$$

The proofs for $n = 3t + 2$ and $n = 3t$ are very similar. □

Our new proof of Theorem 1.1 follows immediately from Lemma 3.5 by slightly modifying the proof of Theorem 3.2.

Proof. First, from weak duality and Lemma 3.5, we have

$$LP(n) \leq LP_f(n).$$

Combining this inequality with (2) and (3), we have

$$N(n) \leq \lfloor LP(n) \rfloor \leq \lfloor LP_f(n) \rfloor = N_f(n).$$

We showed in Theorem 2.1 that $N(n) \geq N_f(n)$; hence $N(n) = N_f(n)$. \square

4 Discussion

We investigated the “dots in triangles problem” due to an application to honeycomb arrays (see [1]). However, we did not realize that the dots in triangles problem had already been solved. Since we did not know the value of $N(n)$, we adopted an “experimental” approach:

1. We used Maple to gather some numerical data.
2. We formulated (obvious) conjectures based on the numerical data.
3. Finally, we proved the conjectures mathematically.

Many problems in combinatorics are amenable to such an approach, but this particular problem serves as an ideal illustration of the usefulness of this methodology. Indeed, the problem seemed almost to “solve itself”, with minimal thought or human ingenuity required!

It should also be emphasized that, in the end, the resulting proof is quite short and simple:

1. By a suitable direct construction, prove that $N(n) \geq \lfloor \frac{2n+1}{3} \rfloor$.
2. Show that the dual LP has a feasible solution whose objective function value is less than $\lfloor \frac{2n+1}{3} \rfloor + 1$.

The first conjecture we posed was the LP Conjecture, concerning the optimal solutions to the LP. In general, to prove a feasible solution to an LP is optimal, it is necessary to do the following:

1. Find a feasible solution to the primal LP and denote the value of the objective function by C .

2. Find a feasible solution to the dual LP and denote the value of the objective function by C^* .

If $C = C^*$, then the solution to the LP is optimal (this is often called *strong duality*).

When $n \equiv 1 \pmod{3}$, our work in fact proves the LP conjecture. This is because Theorem 2.1 yields a solution to the primal LP whose objective function value matches the solution we later found to the dual LP. However, when $n \not\equiv 1 \pmod{3}$, we do not have a general solution to the primal LP whose objective function value matches the solution to the dual LP. Although we are confident that the LP conjecture is also true for these values of n , proving it could get messy!

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