Rainbow Trees in Small Cubic Graphs

Futaba Fujie-Okamoto
Mathematics Department
University of Wisconsin La Crosse
La Crosse, WI 54601
okamoto.futa@uwlax.edu
Jianwei Lin
Department of Mathematics
Western Michigan University
Kalamazoo, MI 49008
jianwei.lin@wmich.edu
Ping Zhang
Department of Mathematics
Western Michigan University
Kalamazoo, MI 49008
ping.zhang@wmich.edu

Abstract

Let G be a nontrivial connected graph of order n and k an integer with $2 \leq k \leq n$. For a set S of k vertices of G, let $\kappa(S)$ denote the maximum number ℓ of pairwise edge-disjoint trees $T_1, T_2, \ldots, T_\ell$ in G such that $V(T_i) \cap V(T_j) = S$ for every pair i, j of distinct integers with $1 \leq i, j \leq \ell$. A collection $\{T_1, T_2, \ldots, T_\ell\}$ of trees in G with this property is called a set of internally disjoint trees connecting S. The k-connectivity $\kappa_k(G)$ of G is defined as $\kappa_k(G) = \min\{\kappa(S)\}$, where the minimum is taken over all k-element subsets S of $V(G)$. Thus $\kappa_2(G)$ is the connectivity $\kappa(G)$ of G. In an edge-colored graph G in which adjacent edges may be colored the same, a tree T is a rainbow tree in G if no two edges of T are colored the same. For each integer ℓ with $1 \leq \ell \leq \kappa_k(G)$, a (k, ℓ)-rainbow coloring of G is an edge coloring of G (in which adjacent edges may be colored the same) such that every set S of k vertices of G has ℓ internally disjoint rainbow trees connecting S. The (k, ℓ)-rainbow index $\text{rx}_{k,\ell}(G)$ of G is the smallest number of colors needed in a (k, ℓ)-rainbow coloring of G. In this work, we investigate the (k, ℓ)-rainbow indices of small cubic graphs.

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1 Introduction

In an edge-colored graph G in which adjacent edges may be colored the same, a path P is a *rainbow path* if no two edges of P are colored the same. The graph G is *rainbow-connected* if G contains a $u - v$ rainbow path for each pair u, v of distinct vertices of G. An edge coloring of a connected graph G that results in a rainbow-connected graph is a *rainbow coloring* of G. If p colors are used in a rainbow coloring, then this coloring is referred to as a *rainbow p-edge coloring*. The minimum p for which G has a rainbow p-edge coloring is the *rainbow connection number* $\mathrm{rc}(G)$ of G. These concepts were introduced and studied in [4] and studied further in [1, 2, 3, 5, 9, 10]. It is known that computing the rainbow connection number of a graph is NP-Hard and determining whether a given edge-colored graph (with an unbounded number of colors) is rainbow connected is NP-Complete (see [2]).

As described in [5, 7], Ericksen [8] stated in 2007 that following the terrorist attacks on September 11, 2001, it was observed that intelligence agencies were not able to communicate with each other through their regular channels from radio systems to databases. Although such information needs to be protected because it is critical to national security, procedures must be in place that permit access between appropriate parties. Ericksen went on to say that this can be addressed by assigning information transfer paths between agencies which may have other agencies as intermediaries, where the number of passwords and firewalls required that are prohibitive to intruders is large enough so that any path between agencies has no passwords repeated yet small enough for smooth communication among agencies. This situation can be modeled by a graph and studied by means of rainbow colorings. Recently, several generalizations of these concepts have been introduced.

If G is a connected graph with connectivity $\kappa(G) = \kappa$, then it follows from a well-known theorem of Whitney [12] that for every integer ℓ with $1 \leq \ell \leq \kappa$ and every two distinct vertices u and v of G, the graph G contains ℓ internally disjoint $u - v$ paths. A generalization of rainbow-connected graphs based on Whitney's theorem was introduced in [5]. For a connected graph G and an integer ℓ with $1 \leq \ell \leq \kappa(G)$, the *rainbow ℓ-connectivity* $\mathrm{rc}_\ell(G)$ is the minimum number of colors needed in an edge coloring of G (where adjacent edges may be colored the same) such that for every two distinct vertices u and v of G, there exist at least ℓ internally disjoint $u - v$ rainbow paths. Thus $\mathrm{rc}_1(G) = \mathrm{rc}(G)$. When $\ell = \kappa(G)$, the rainbow ℓ-connectivity of G is simply referred to as the *rainbow connectivity* $\kappa_r(G)$ of G. These concepts were introduced and studied in [5] and studied further in [10].

For example, the connectivity of the 3-cube Q_3 is $\kappa(Q_3) = 3$. Rainbow p-edge colorings of Q_3 are shown in Figure 1 for $p = 3, 4, 7$. In the 3-edge-colored graph shown in Figure 1(a), every two distinct vertices are connected by a rainbow path; in the 4-edge-colored graph in Figure 1(b), there exist at least two internally disjoint $u - v$ rainbow paths for every two distinct vertices u and v; while in the 7-edge-colored graph in Figure 1(c), there exist at least three internally disjoint $u - v$ rainbow paths for every two distinct vertices u and v. In fact, $\mathrm{rc}_1(Q_3) = \mathrm{rc}(Q_3) = 3$, $\mathrm{rc}_2(Q_3) = 4$ and $\mathrm{rc}_3(Q_3) = \kappa_r(Q_3) = 7$ (see [10]).

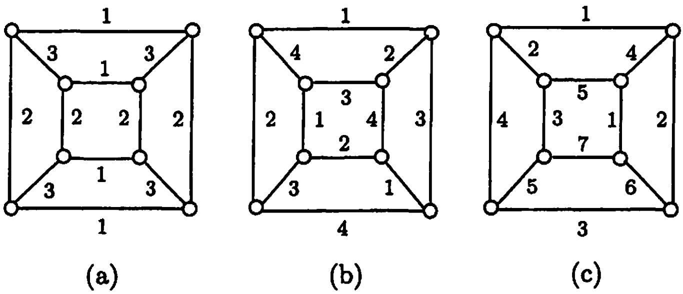

Figure 1: Rainbow p-edge colorings of Q_3 for $p = 3, 4, 7$

In [7] the concept of rainbow colorings was extended to involve rainbow trees in graphs. Let G be a nontrivial connected graph of order n on which is defined an edge coloring where adjacent edges may be assigned the same color. A tree T in G is a *rainbow tree* if no two edges of T are colored the same. For an integer k with $2 \le k \le n$, an edge coloring of G is called a *k-rainbow tree coloring* (or simply *a k-rainbow coloring*) if for every set S of k vertices of G, there exists a rainbow tree in G containing the vertices of S. The *k-rainbow index* $\mathrm{rx}_k(G)$ of G is the minimum number of colors needed in a k-rainbow coloring of G. Thus $\mathrm{rx}_2(G)$ is the rainbow connection number $\mathrm{rc}(G)$ of G. Furthermore,

$$\mathrm{rx}_2(G) \le \mathrm{rx}_3(G) \le \cdots \le \mathrm{rx}_n(G)$$

for every nontrivial connected graph G of order n. Rainbow trees and rainbow indices of graphs were introduced and studied in [7].

For example, consider the graph G of Figure 2, where k-rainbow colorings of G are shown for $k = 2, 3, 4$. In the 5-edge-colored graph shown in Figure 2(a), every two distinct vertices are connected by a rainbow path; in the 6-edge-colored graph in Figure 2(b), every three distinct vertices are contained in a rainbow tree; while in the 7-edge-colored graph in Figure 2(c), every four distinct vertices are contained in a rainbow tree. In fact, $rx_2(G) = rc(G) = 5$, $rx_3(G) = 6$ and $rx_k(G) = 7$ for $4 \leq k \leq 8$ (see [7]).

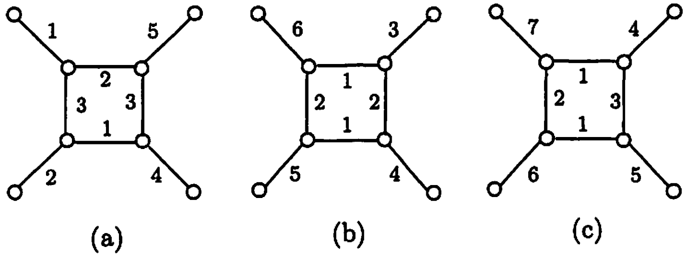

Figure 2: k-Rainbow colorings of a graph for $k = 5, 6, 7$

The concepts and problems concerning rainbow trees gave rise to a generalized connectivity in [7] using trees with the aid of factorizations. Let G be a nontrivial connected graph of order n and k an integer with $2 \leq k \leq n$. For a set S of k vertices of G, let $\kappa(S)$ denote the maximum number ℓ of pairwise edge-disjoint trees $T_1, T_2, \ldots, T_\ell$ in G such that $V(T_i) \cap V(T_j) = S$ for every pair i, j of distinct integers with $1 \leq i, j \leq \ell$. A collection $\{T_1, T_2, \ldots, T_\ell\}$ of trees in G with this property is called a *set of internally disjoint trees connecting* S. The *k-connectivity* $\kappa_k(G)$ of G is defined as $\kappa_k(G) = \min\{\kappa(S)\}$, where the minimum is taken over all k-element subsets S of $V(G)$. Thus $\kappa_2(G)$ is the connectivity $\kappa(G)$. These concepts were introduced and studied in [7] and studied further in [11].

For example, consider the complete 3-partite graph $G = K_{3,4,5}$ whose partite sets are $U = \{u_1, u_2, u_3\}$, $V = \{v_1, v_2, v_3, v_4\}$ and $W = \{w_1, w_2, w_3, w_4, w_5\}$. Then $\kappa(G) = \kappa_2(G) = 7$. For $S = \{u_1, v_1, w_1\}$, the graph G contains the six internally disjoint trees of Figure 3 connecting S. Furthermore, it can be shown that for each 3-element subset S of $V(G)$, we have $6 \leq \kappa_3(S) \leq 9$. Therefore, $\kappa_3(G) = 6$ (see [7]).

This generalized connectivity of a graph was extended in [7] to edge-colored graphs using rainbow trees. For a nontrivial connected graph G of order n, let k and ℓ be integers with $2 \leq k \leq n$ and $1 \leq \ell \leq \kappa_k(G)$. A *$(k, \ell)$-rainbow coloring* of G is an edge coloring of G (where adjacent edges may be assigned the same color) such that every set S of k vertices of G has ℓ internally disjoint rainbow trees connecting S. The *(k, ℓ)-rainbow index* $rx_{k,\ell}(G)$ of G is the smallest number of colors needed in a (k, ℓ)-rainbow coloring of G. Therefore, $rx_{k,1}(G) = rx_k(G)$ and $rx_{2,\ell}(G) = rc_\ell(G)$. In particular, $rx_{2,1}(G) = rx_2(G) = rc(G)$ and $rx_{2,\kappa(G)}(G)$ is the rainbow

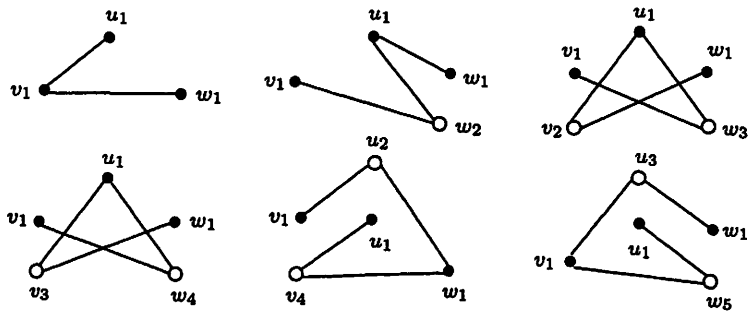

Figure 3: Six internally disjoint trees in $K_{3,4,5}$ connecting $\{u_1, v_1, w_1\}$

connectivity $\kappa_r(G)$. These concepts were introduced and studied in [7].

For example, it is known that $\kappa_3(K_6) = 4$ (see [7]). Thus, we consider $rx_{3,\ell}(K_6)$ for $1 \le \ell \le 4$. In the 3-edge-colored graph shown in Figure 4(a), for every set S of three vertices, there exist three internally disjoint rainbow trees connecting S; while in the 4-edge-colored graph in Figure 4(b), there exist four internally disjoint rainbow trees connecting S. In fact, $rx_{3,1}(K_6) = rx_{3,2}(K_6) = rx_{3,3}(K_6) = 3$ and $rx_{3,4}(K_6) = 4$ (see [7]).

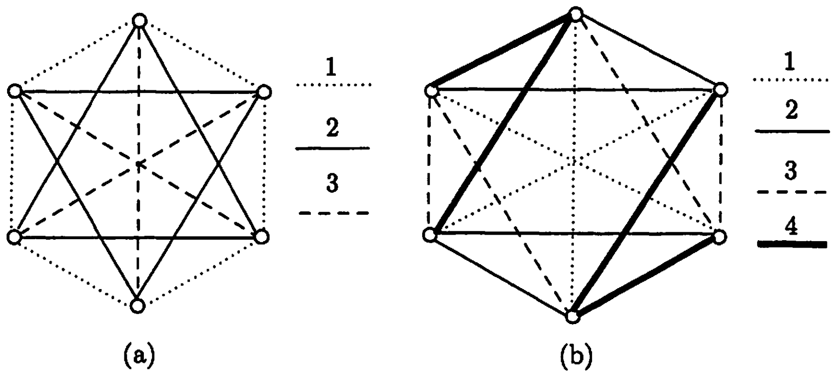

Figure 4: A rainbow 3-edge coloring and a rainbow 4-edge coloring of K_6

In [10] the rainbow ℓ-connectivities of all cubic graphs of order at most 8 and of the Petersen graph were determined for $1 \le \ell \le 3$. In this work, we investigate the rainbow indices of three well-known small cubic graphs, namely the complete bipartite graph $K_{3,3}$, the Cartesian product $K_3 \times K_2$ and the 3-cube Q_3. We refer to the book [6] for any graph theory notation and terminology not described in this paper.

2 Some Observations on the Rainbow Indices of Cubic Graphs

In this section, we present several useful results on the k-rainbow index and (k,ℓ)-rainbow index of a graph. For vertices u and v in a nontrivial connected graph G, the *distance* $d(u,v)$ between u and v is the length of a shortest $u-v$ path in G. The *diameter* $\mathrm{diam}(G)$ of G is the greatest distance between two vertices of G. For a set S of vertices in G, the *Steiner distance* $d(S)$ of S is the minimum size of a tree in G containing S. Such a tree is called a *Steiner S-tree* or simply a *Steiner tree*. The *k-Steiner diameter*, $\mathrm{sdiam}_k(G)$, of G is the maximum Steiner distance among all sets of k vertices in G. Thus if $k=2$ and $S=\{u,v\}$, then $d(S)=d(u,v)$ and the 2-Steiner diameter $\mathrm{sdiam}_2(G)$ simply equals $\mathrm{diam}(G)$. There are some immediate upper and lower bounds for the k-rainbow index $\mathrm{rx}_k(G)$ in terms of the order and Steiner diameter of G, regardless of the value of k. We begin with a result stated in [7].

Proposition 2.1 [7] *Let G be a connected graph of order $n \geq 4$. For each integer k with $3 \leq k \leq n-1$,*

$$k-1 \leq \mathrm{sdiam}_k(G) \leq \mathrm{rx}_k(G) \leq n-1 \quad \textit{and} \quad \mathrm{rx}_n(G) = n-1.$$

Next, we make two useful observations.

Observation 2.2 *Let G be an r-regular connected graph of order $n \geq 4$. For each integer k with $3 \leq k \leq n$,*

$$\kappa_k(G) \leq r-1.$$

In particular, if G is a cubic graph, then $\kappa_k(G) \leq 2$.

Observation 2.3 *For a connected graph G of order $n \geq 4$, an integer k with $3 \leq k \leq n$ and an integer ℓ with $1 \leq \ell \leq \kappa_k(G)$,*

$$\mathrm{rx}_{k,\ell}(G) \geq \mathrm{sdiam}_k(G) \geq k-1. \tag{1}$$

Furthermore, if the size of G is m, then

$$\ell(k-1) \leq m. \tag{2}$$

By (2) in Observation 2.3, if G is a connected cubic graph of order n, then $\kappa_n(G)=1$ for $n \geq 6$ and $\kappa_{n-1}(G)=1$ for $n \geq 10$. That is, for each integer $\ell \geq 2$, $\mathrm{rx}_{n,\ell}(G)$ is undefined for $n \geq 6$ and $\mathrm{rx}_{n-1,\ell}(G)$ is undefined for $n \geq 10$. In particular, $\mathrm{rx}_{n,2}(G)$ is undefined for $n \geq 6$ and $\mathrm{rx}_{n-1,2}(G)$ is undefined for $n \geq 10$. The following result will be useful to us.

Proposition 2.4 *If G is a connected cubic graph of order n whose $(n-1)$-connectivity is 2, then $n \leq 6$ and $\mathrm{rx}_{n-1,2}(G) \geq n-1$.*

Proof. Since $\kappa_{n-1}(G) = 2$, there is an $(n-1, 2)$-rainbow coloring of G. Let S be an $(n-1)$-element subset of $V(G)$, where say $S = V(G) - \{v\}$. Let T_1 and T_2 be two internally disjoint rainbow trees connecting S. Assume, to the contrary, that $\mathrm{rx}_{n-1,2}(G) \leq n-2$. Then both T_1 and T_2 must be of size $n-2$ and so $V(T_1) = V(T_2) = S$. Then none of the edges incident with v belong to the two trees. Since the size of G is $\frac{3}{2}n$, it follows that

$$\tfrac{3}{2}n - 3 \geq |E(T_1)| + |E(T_2)| = 2(n-2).$$

However, this in turn implies that $n \leq 2$, which is impossible. Therefore, $\mathrm{rx}_{n-1,2}(G) \geq n-1$ and we may assume that $V(T_2) = V(G)$. Hence $\frac{3}{2}n \geq |E(T_1)| + |E(T_2)| > 2(n-2)$. Since the order must be even, $n \leq 6$. ■

By Proposition 2.4, if G is a cubic graph of order $n \geq 8$, then $\mathrm{rx}_{n-1,2}(G)$ is undefined.

3 Rainbow Indices of Three Small Cubic Graphs

By Observation 2.2, if G is a connected cubic graph, then $\mathrm{rx}_{k,\ell}(G)$ is undefined for $\ell = 3$ and $k \geq 3$. Also, we have seen that $\mathrm{rc}_\ell(G) = \mathrm{rx}_{2,\ell}(G)$ for each ℓ, where $\mathrm{rc}_\ell(G)$ is the rainbow ℓ-connectivity of G. In [10] the rainbow ℓ-connectivities of cubic graphs of order less than or equal to 8 and the Petersen graph were determined for $1 \leq \ell \leq 3$. Among the results obtained are the following:

- $\mathrm{rc}_1(K_{3,3}) = 2$, $\mathrm{rc}_2(K_{3,3}) = 3$ and $\mathrm{rc}_3(K_{3,3}) = 3$.
- $\mathrm{rc}_1(K_3 \times K_2) = 2$, $\mathrm{rc}_2(K_3 \times K_2) = 3$ and $\mathrm{rc}_3(K_3 \times K_2) = 6$.
- $\mathrm{rc}_1(Q_3) = 3$, $\mathrm{rc}_2(Q_3) = 4$ and $\mathrm{rc}_3(Q_3) = 7$.

In this section, we consider these three small cubic graphs and their (k, ℓ)-rainbow indices for all possible pairs k and ℓ, where $3 \leq k \leq n$ and $\ell = 1, 2$.

3.1 Rainbow Indices of $K_{3,3}$

We first determine $\mathrm{rx}_{k,\ell}(K_{3,3})$ for all possible values of $k \geq 3$ and $\ell = 1, 2$. Observe that $\mathrm{rx}_{6,2}(K_{3,3})$ is undefined by (2). We assume that the partite sets of $K_{3,3}$ are $U = \{u_1, u_2, u_3\}$ and $W = \{w_1, w_2, w_3\}$. Figure 5 shows three rainbow edge colorings of $K_{3,3}$.

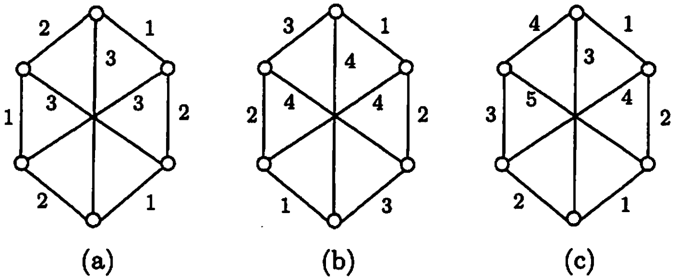

Figure 5: Three rainbow edge colorings of $K_{3,3}$

Proposition 3.1 *For* $G = K_{3,3}$, $\mathrm{rx}_{3,1}(G) = \mathrm{rx}_{3,2}(G) = 3$ *and* $\mathrm{rx}_{k,1}(G) = k-1$ *for* $4 \le k \le 6$.

Proof. We first show that $\mathrm{rx}_{3,1}(G) = \mathrm{rx}_{3,2}(G) = 3$. Since the Steiner distance of each partite set is 3, it follows by (1) that $\mathrm{rx}_{3,\ell}(G) \ge 3$ for $\ell = 1,2$. On the other hand, a proper 3-edge coloring of G (see Figure 5(a)) shows that $\mathrm{rx}_{3,\ell}(G) \le 3$ for $\ell = 1,2$.

Next, we show that $\mathrm{rx}_{k,1}(G) = k-1$ for $4 \le k \le 6$. With the aid of Proposition 2.1 and (1), we need only show that $\mathrm{rx}_{k,1}(G) \le k-1$ for $k = 4,5$. The 3-edge coloring given in Figure 5(a) is a 4-rainbow coloring; while the 4-edge coloring given in Figure 5(b) is a 5-rainbow coloring. Therefore, $\mathrm{rx}_{k,1}(G) = k-1$ for $4 \le k \le 6$. ■

Theorem 3.2 *For* $G = K_{3,3}$, $\mathrm{rx}_{4,2}(G) = \mathrm{rx}_{5,2}(G) = 5$.

Proof. The coloring in Figure 5(c) shows that $\mathrm{rx}_{k,2}(G) \le 5$ for $k = 4,5$. Then $\mathrm{rx}_{5,2}(G) = 5$ by Proposition 2.4.

Assume, to the contrary, that $\mathrm{rx}_{4,2}(G) \le 4$. Let c be a $(4,2)$-rainbow coloring of G using the colors in $\{1,2,3,4\}$. Then at least three edges of G are assigned the same color, say there are three edges colored 1. Let $S = \{u_1, u_2, w_1, w_2\}$ and suppose that $\mathcal{T} = \{T_1, T_2\}$ is a set of internally disjoint rainbow trees connecting S. Since the size of every rainbow tree is at most 4, each tree in $\mathcal{T}$ contains at most one of u_3 and w_3. Therefore, every vertex has degree 1 or 2 in each tree. If $T_1 \cong P_4$, say, then we may assume that $T_1 = (u_1, w_1, u_2, w_2)$. However, this implies that both w_3 and u_3 belong to T_2, a contradiction. Thus $T_1 \cong T_2 \cong P_5$ and furthermore, $T_1 = (u_1, w_{i_1}, u_3, w_{i_2}, u_2)$ and $T_2 = (w_{i_1}, u_2, w_3, u_1, w_{i_2})$, where $\{i_1, i_2\} = \{1,2\}$, is the only possibility. Therefore, $c(u_1w_{i_1}) \ne c(u_2w_{i_2})$ and $c(u_1w_{i_2}) \ne c(u_2w_{i_1})$, which in turn, by considering all possibilities of

S, implies that every two independent edges must be assigned distinct colors. Since there are at least three edges colored 1, it follows that there is a monochromatic $K_{1,3}$, say $c(u_1w_1) = c(u_1w_2) = c(u_1w_3) = 1$. However then, for $S' = \{u_2, w_1, w_2, w_3\}$, there are no two internally disjoint rainbow trees connecting S'. Therefore, $\text{rx}_{4,2}(G) \geq 5$. ∎

Table 1 summarizes the numbers $\text{rx}_{k,\ell}(K_{3,3})$ for all possible values of k and ℓ, where the dashed lines "–" indicate the corresponding numbers do not exist.

	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
$\ell=1$	$\text{rx}_{2,1}(G)=2$	$\text{rx}_{3,1}(G)=3$	$\text{rx}_{4,1}(G)=3$	$\text{rx}_{5,1}(G)=4$	$\text{rx}_{6,1}(G)=5$
$\ell=2$	$\text{rx}_{2,2}(G)=3$	$\text{rx}_{3,2}(G)=3$	$\text{rx}_{4,2}(G)=5$	$\text{rx}_{5,2}(G)=5$	—
$\ell=3$	$\text{rx}_{2,3}(G)=3$	—	—	—	—

Table 1: The numbers $\text{rx}_{k,\ell}(G)$ for $G = K_{3,3}$

3.2 Rainbow Indices of $K_3 \times K_2$

We next determine $\text{rx}_{k,\ell}(K_3 \times K_2)$ for all possible values of $k \geq 3$ and $\ell = 1, 2$. Again $\text{rx}_{6,2}(K_3 \times K_2)$ is undefined by (2). Let $G = K_3 \times K_2$ with $V(G) = U \cup W$, where the three vertices in each of the two disjoint sets $U = \{u_1, u_2, u_3\}$ and $W = \{w_1, w_2, w_3\}$ form a triangle and $u_iw_i \in E(G)$ for $1 \leq i \leq 3$ (see Figure 6). Furthermore, let $X_U = \{u_1u_2, u_1u_3, u_2u_3\}$, $X_W = \{w_1w_2, w_1w_3, w_2w_3\}$ and $Y = \{u_1w_1, u_2w_2, u_3w_3\}$. Figure 6 shows three rainbow edge colorings of $K_3 \times K_2$.

Proposition 3.3 *For $G = K_3 \times K_2$, $\text{rx}_{3,1}(G) = \text{rx}_{4,1}(G) = 3$.*

Proof. Since the Steiner distance of the set $\{u_1, u_2, w_3\}$ is 3, it follows that $\text{rx}_{4,1}(G) \geq \text{rx}_{3,1}(G) \geq 3$; while the 3-edge coloring of G in Figure 6(a) shows that $\text{rx}_{3,1}(G) \leq \text{rx}_{4,1}(G) \leq 3$. ∎

Proposition 3.4 *For $G = K_3 \times K_2$, $\text{rx}_{5,1}(G) = \text{rx}_{3,2}(G) = 4$.*

Proof. The 4-edge coloring of G in Figure 6(b) shows that the two numbers are at most 4. Then $\text{rx}_{5,1}(G) = 4$ by (1). To see that $\text{rx}_{3,2}(G) \geq 4$, suppose that there is a $(3,2)$-rainbow coloring of G using three colors. If T is a tree of order at most 4 whose vertex set contains U, then two of the three edges in X_U must belong to T. Since any rainbow tree must be of order at most 4, there is only one rainbow tree connecting U, which is a contradiction. Therefore, $\text{rx}_{3,2}(G) \geq 4$ and so $\text{rx}_{3,2}(G) = 4$. ∎

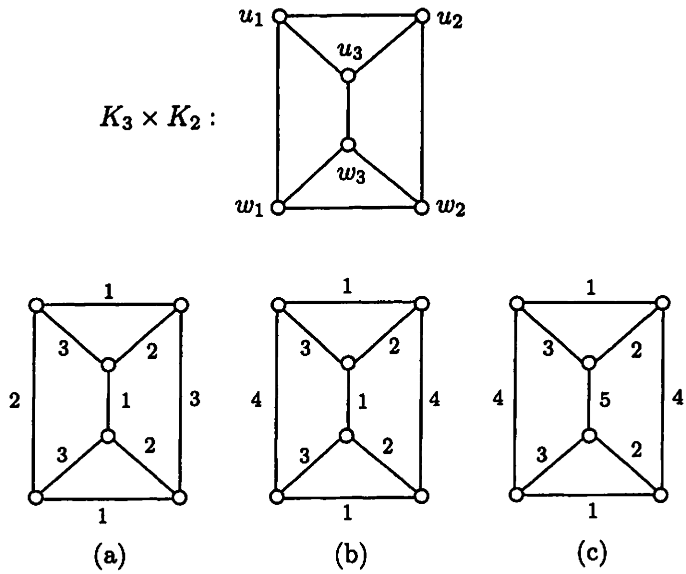

Figure 6: The graph $K_3 \times K_2$ and three rainbow edge colorings

Theorem 3.5 *For* $G = K_3 \times K_2$, $\mathrm{rx}_{6,1}(G) = \mathrm{rx}_{4,2}(G) = \mathrm{rx}_{5,2}(G) = 5$.

Proof. By Proposition 2.1, $\mathrm{rx}_{6,1}(G) = 5$. Also, the 5-edge coloring of G in Figure 6(c) shows that $\mathrm{rx}_{k,2}(G) \leq 5$ for $k = 4, 5$. Therefore, $\mathrm{rx}_{5,2}(G) = 5$ by Proposition 2.4.

Assume, to the contrary, that $\mathrm{rx}_{4,2}(G) \leq 4$. Let c be a $(4,2)$-rainbow coloring of G using four colors. We first verify two claims:

(A) There is no monochromatic triangle.

(B) If e and f are adjacent edges and $c(e) = c(f)$, then $\{e, f\} \cap Y = \emptyset$.

If (A) is false, then suppose that the three edges in X_U are assigned the same color. Then every rainbow tree containing U contains at most one of the three edges in X_U. Therefore, if T is a rainbow tree containing U, then it must contain at least two of the three edges in Y. This implies that it is impossible to find two edge-disjoint rainbow trees each of which contains U.

Next, suppose that (B) is false. Without loss of generality, suppose that $c(u_1w_1) = c(w_1w_3)$ and consider the set $S = U \cup \{w_3\}$. Suppose that T_1

and T_2 are internally disjoint trees of order at most 5 connecting S such that $u_3w_3 \notin E(T_1)$. If T_1 is a rainbow tree, then $V(T_1) = S \cup \{w_2\}$ and the edges u_2w_2 and w_2w_3 must belong to T_1. Moreover, two edges in X_U belong to T_1. However, this implies that it is impossible for T_2 to be a rainbow tree, which is a contradiction.

Therefore, as claimed, (A) and (B) hold. Let E_1, E_2, E_3 and E_4 be the color classes and without loss of generality, suppose that $|E_1| \geq \lceil \frac{9}{4} \rceil = 3$. We consider two cases.

Case 1. *$Y \cap E_1 \neq \emptyset$, say $c(u_1w_1) = 1$.* Then by (B) either (i) $E_1 = Y$ or (ii) $E_1 = \{u_1w_1, u_2u_3, w_2w_3\}$. If (i) occurs, then suppose that T_1 and T_2 are edge-disjoint trees containing U. At most one of T_1 and T_2 can contain two edges in X_U and so suppose that $|E(T_1) \cap X_U| \leq 1$. Then T_1 must contain two edges in Y, implying that T_1 is not a rainbow tree. Therefore, it is impossible to find two internally disjoint rainbow trees connecting S, where $U \subseteq S$.

If (ii) occurs, then suppose that T_1 and T_2 are internally disjoint rainbow trees connecting the set $S = U \cup \{w_3\}$ and without loss of generality, $u_3w_3 \notin E(T_1)$. Then either (iia) $u_1w_1, w_1w_3 \in E(T_1)$ or (iib) $u_2w_2, w_2w_3 \in E(T_1)$. If (iia) occurs, then $u_1u_2, u_1u_3 \in E(T_1)$. Since the three edges incident with u_1 all belong to T_1, it follows that T_2 does not contain u_1, which is impossible. If (iib) occurs, on the other hand, then $u_1u_2, u_1u_3 \in E(T_1)$. Then every tree T containing u_1 and u_2 that is edge-disjoint from T_1 has to contain the edges u_1w_1 and u_2u_3, which implies that T cannot be a rainbow tree. This is impossible, too. Therefore, (ii) does not occur either.

Case 2. *$Y \cap E_1 = \emptyset$.* Then we may assume that $u_1u_3, u_2u_3 \in E_1$. Since $u_1u_2 \notin E_1$ by (A), there is another edge belonging to X_W that is assigned the color 1. However, a similar argument used in Case 1(ii) shows that it is impossible to find two internally disjoint rainbow trees connecting the set $S = W \cup \{u_3\}$, which is impossible.

Therefore, $\mathrm{rx}_{4,2}(G) = 5$, as claimed. ∎

Table 2 summarizes the numbers $\mathrm{rx}_{k,\ell}(K_3 \times K_2)$, where the dashed lines "–" indicate the corresponding numbers do not exist.

	$k = 2$	$k = 3$	$k = 4$	$k = 5$	$k = 6$
$\ell = 1$	$\mathrm{rx}_{2,1}(G) = 2$	$\mathrm{rx}_{3,1}(G) = 3$	$\mathrm{rx}_{4,1}(G) = 3$	$\mathrm{rx}_{5,1}(G) = 4$	$\mathrm{rx}_{6,1}(G) = 5$
$\ell = 2$	$\mathrm{rx}_{2,2}(G) = 3$	$\mathrm{rx}_{3,2}(G) = 4$	$\mathrm{rx}_{4,2}(G) = 5$	$\mathrm{rx}_{5,2}(G) = 5$	—
$\ell = 3$	$\mathrm{rx}_{2,3}(G) = 6$	—	—	—	—

Table 2: The numbers $\mathrm{rx}_{k,\ell}(G)$ for $G = K_3 \times K_2$

3.3 Rainbow Indices of Q_3

We now consider the 3-cube Q_3. Suppose that the vertices of Q_3 are labeled as shown in Figure 7, where then $U = \{u_1, u_2, u_3, u_4\}$ and $W = \{w_1, w_2, w_3, w_4\}$ are the partite sets.

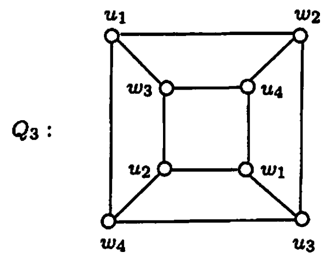

Figure 7: The 3-cube Q_3

By Proposition 2.4 and (2), $\mathrm{rx}_{k,2}(Q_3)$ is undefined for $k = 7, 8$. Figure 8 shows three rainbow edge colorings of Q_3.

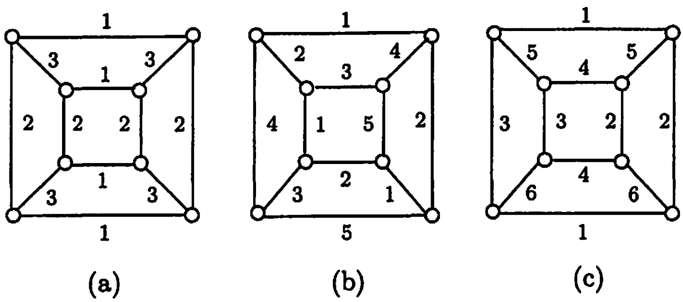

Figure 8: Three rainbow edge colorings of Q_3

Theorem 3.6 *For* $G = Q_3$, $\mathrm{rx}_{3,1}(G) = 3$, $\mathrm{rx}_{4,1}(G) = \mathrm{rx}_{5,1}(G) = 5$, $\mathrm{rx}_{6,1}(G) = \mathrm{rx}_{7,1}(G) = 6$ *and* $\mathrm{rx}_{8,1}(G) = 7$.

Proof. That $\mathrm{rx}_{8,1}(G) = 7$ is immediate by Proposition 2.1. To see that $\mathrm{rx}_{3,1}(G) = 3$, the coloring in Figure 8(a) shows that $\mathrm{rx}_{3,1}(G) \leq 3$; while the Steiner distance of the set $S = \{u_1, u_2, u_3\}$ is 3 and so $\mathrm{rx}_{3,1}(G) \geq 3$ by (1).

For $k = 4, 5$, the coloring in Figure 8(b) shows that $\mathrm{rx}_{k,1}(G) \leq 5$; while the Steiner distance $d(S)$ is 5 for $S \in \{U, U \cup \{w_1\}\}$ and so it follows by (1) that $\mathrm{rx}_{k,1}(G) \geq 5$. Therefore, $\mathrm{rx}_{4,1}(G) = \mathrm{rx}_{5,1}(G) = 5$.

For $k = 6, 7$, Figure 8(c) shows that $\text{rx}_{k,1}(G) \le 6$ and so $\text{rx}_{7,1}(G) = 6$ by (1). If $\text{rx}_{6,1}(G) = 5$, then let c be a 5-rainbow coloring of G using five colors and, without loss of generality, suppose that E_1 is a color class with $|E_1| \le 2 = \lfloor \frac{12}{5} \rfloor$. If $E_1 = \{e = uw, e' = u'w'\}$, where $u, u' \in U$ and $w, w' \in W$, then consider the set $S = V(G) - \{u, w'\}$. By assumption, there exists a rainbow tree T connecting S. Furthermore, T must be of size 5 and contain one of the edges e and e'. However then, one of the vertices u and w' must belong to T, implying that T contains more than 6 vertices, which is a contradiction. If $E_1 = \{e = uw\}$, then consider the set $S = V(G) - \{u, w\}$ and a similar argument shows that there is no rainbow tree connecting S. Therefore, $\text{rx}_{6,1}(G) \ge 6$ and the desired result follows. ∎

Theorem 3.7 *For* $G = Q_3$, $\text{rx}_{4,2}(G) = 6$.

Proof. Since the Steiner distance of the set U is 5, we see that $\text{rx}_{4,2}(G) \ge 5$. On the other hand, the coloring in Figure 8(c) shows that $\text{rx}_{4,2}(G) \le 6$. Assume, to the contrary, that there exists a $(4, 2)$-rainbow coloring c of G using five colors. Consider the four cycle $C = (u_1, w_2, u_3, w_4, u_1)$ and let T_1 and T_2 be internally disjoint trees connecting the set $V(C)$. Observe that each of the four edges in C belongs to either T_1 or T_2. Furthermore, if e and f are two edges in C that are not adjacent, then e and f belong to different trees. Therefore, we may assume that $u_1w_2, u_3w_2 \in E(T_1)$ and $u_1w_4, u_3w_4 \in E(T_2)$. Then $u_2w_4 \in E(T_1)$ and $u_4w_2 \in E(T_2)$ and so we may further assume that $u_2w_1, u_3w_1 \in E(T_1)$ and $u_1w_3, u_4w_3 \in E(T_2)$. Therefore, both T_1 and T_2 are of size 5, implying that $|E_i| \ge 2$ for each color class E_i $(1 \le i \le 5)$. Also, the two edges u_2w_3 and u_4w_1 belong to neither T_1 nor T_2. Therefore, if the 4-cycle formed by the vertices in the set $V(G) - V(C)$ is denoted by C', then C' contains two edges that are not adjacent and belong to neither of two internally disjoint rainbow trees connecting $V(C)$. Without loss of generality, suppose that $|E_1| \ge |E_2| \ge \cdots \ge |E_5|$. Then either (i) $|E_1| = 4$ and $|E_i| = 2$ for $2 \le i \le 5$ or (ii) $|E_1| = |E_2| = 3$ and $|E_i| = 2$ for $3 \le i \le 5$. If (i) occurs, then $c(u_2w_3) = c(u_4w_1) = 1$. Also, since C was arbitrarily chosen, every 4-cycle contains two nonadjacent edges e and f with $c(e) = c(f) = 1$. However, this is impossible since $|E_1| = 4$. If (ii) occurs, then $\{c(u_2w_3), c(u_4w_1)\} = \{1, 2\}$ and, in general, every 4-cycle contains two nonadjacent edges e and f such that $\{c(e), c(f)\} = \{1, 2\}$. Again this is impossible. We conclude that $\text{rx}_{4,2}(G) \ge 6$ and so $\text{rx}_{4,2}(G) = 6$. ∎

Theorem 3.8 *For* $G = Q_3$, $\text{rx}_{6,2}(G) = 12$.

Proof. Let there be given a $(6, 2)$-rainbow coloring c of $G = Q_3$. Consider the set $S = V(G) - \{u_1, w_2\}$ and suppose that T_1 and T_2 are internally disjoint rainbow trees connecting S. If u_1 belongs to neither T_1 nor T_2, then $10 \le |E(T_1)| + |E(T_2)| \le |E(G)| - \deg u_1 = 9$, which is clearly impossible. Hence, each of u_1 and w_2 belongs to one of the trees T_1 and T_2. If both T_1 and T_2 are of order 7, then we may assume that $u_1 \in V(T_1)$ and $w_2 \in V(T_2)$. However, this implies that the edge u_1w_2 belongs to neither T_1 nor T_2. Therefore, $11 = |E(G) - \{u_1w_2\}| \ge |E(T_1)| + |E(T_2)| = 12$, a contradiction. Therefore, we assume that $V(T_1) = S$ while $V(T_2) = V(G)$ and $E(G) = E(T_1) \cup E(T_2)$. Hence, the five edges incident with u_1 or w_2 belong to T_2 and are assigned distinct colors. In general, every five edges that form a subgraph isomorphic to a double star of order 6 must be assigned distinct colors. Therefore, if there are two edges e and f with $c(e) = c(f)$, then we may assume, without loss of generality, that $e = u_1w_2$ and $f = u_2w_1$. Now consider the set $S' = V(G) - \{u_3, w_4\}$ and let T_1' and T_2' be internally disjoint rainbow trees connecting S'. We have seen that the five edges incident with u_3 or w_4 must belong to one of the two trees, say T_1'. However then, either T_1' is not a tree or T_2' is not a rainbow tree, a contradiction. Therefore, no two edges can be assigned the same color. ■

Although the exact values of the remaining two rainbow indices of Q_3 (namely $rx_{3,2}(Q_3)$ and $rx_{5,2}(Q_3)$) are unknown, it can be verified that $5 \le rx_{3,2}(Q_3) \le 6 \le rx_{5,2}(Q_3) \le 8$. Table 3 summarizes the numbers $rx_{k,\ell}(Q_3)$.

	$k=2$	$k=3$	$k=4$	$k=5$
$\ell=1$	$rx_{2,1}(G)=3$	$rx_{3,1}(G)=3$	$rx_{4,1}(G)=5$	$rx_{5,1}(G)=5$
$\ell=2$	$rx_{2,2}(G)=4$	$5 \le rx_{3,2}(G) \le 6$	$rx_{4,2}(G)=6$	$6 \le rx_{5,2}(G) \le 8$
$\ell=3$	$rx_{2,3}(G)=7$	—	—	—
	$k=6$	$k=7$	$k=8$	
$\ell=1$	$rx_{6,1}(G)=6$	$rx_{7,1}(G)=6$	$rx_{8,1}(G)=7$	
$\ell=2$	$rx_{6,2}(G)=12$	—	—	
$\ell=3$	—	—	—	

Table 3: The numbers $rx_{k,\ell}(G)$ for $G = Q_3$

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