On the Irreducible No-hole $L(2,1)$ Coloring of Bipartite Graphs and Cartesian Products

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Abstract

The channel assignment problem is the problem of assigning radio frequencies to transmitters while avoiding interference. This problem can be modeled and examined using graphs and graph colorings. $L(2,1)$ coloring was first studied by Griggs and Yeh [6] as a model of a variation of the channel assignment problem. A no-hole coloring, introduced in [4], is defined to be an $L(2,1)$ coloring of a graph which uses all the colors $\{0,1,\ldots,k\}$ for some integer k. An $L(2,1)$ coloring is irreducible, introduced in [3], if no vertex labels in the graph can be decreased and yield another $L(2,1)$ coloring. A graph G is inh-colorable if there exists an irreducible no-hole coloring on G.

We consider the inh-colorability of bipartite graphs and Cartesian products. We obtain some sufficient conditions for bipartite graphs to be inh-colorable. We also find the optimal inh-coloring for some Cartesian products, including grid graphs and the rook's graph.

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1 Introduction

Graph coloring is a well-studied and fertile topic in graph theory. Motivated by the problems from real world applications, many interesting variations and generalizations of graph coloring have evolved. For a survey see [9]. One such variation is motivated by the channel assignment problem in wireless communication.

The channel assignment problem is the problem of assigning radio frequencies to transmitters while avoiding interference. This problem can be modeled and examined using graphs and graph colorings. $L(2,1)$ coloring was first studied by Griggs and Yeh [6] as a model of a variation of the channel assignment problem. An $L(2,1)$ coloring of a graph G is an integer labeling of the vertices where adjacent vertices differ in label by at least two, and vertices that are at distance two from each other differ in label by at least one. That is, an $L(2,1)$ *coloring* of G is a vertex labelling $f : V(G) \longrightarrow \{0\} \cup \mathbb{Z}^+$ such that

1. $|f(u) - f(v)| \geq 2$ for all $uv \in E(G)$,
2. $|f(u) - f(v)| \geq 1$ if $d(u,v) = 2$.

The *span of an* $L(2,1)$ *coloring* f on a graph G is the $\max f(u)$ for all $u \in V(G)$. The *span of a graph* G, denoted by $\lambda(G)$, is the minimum span of all $L(2,1)$ colorings on G. An $L(2,1)$ coloring on G whose span is equal to the span of G is called a *span coloring* of G.

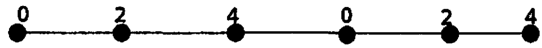
A *full coloring*, introduced in [5], is an $L(2,1)$ coloring that uses every label $\{0, 1, \dots, \lambda(G)\}$. There are many graphs that do not have a full coloring. For example, a complete graph K_n on $n > 1$ vertices does not have a full coloring. A relaxation of full colorings was introduced in [4]. An $L(2,1)$ coloring f on a graph G is a *no-hole* coloring if $f : V(G) \longrightarrow \{0, 1, 2, ..., k\}$ is onto for some k. For example, C_6 does not have a full coloring, but has a no-hole coloring, as shown in Figure 3. A sufficient condition for a graph to be no-hole colorable is provided in [4].

Theorem 1. [4] *Every graph G with $n \geq \lambda(G) + 1$ is no-hole colorable.*

The irreducibility of an $L(2,1)$ coloring was introduced in [3]. An $L(2,1)$ coloring is irreducible if no vertex labels in the graph can be decreased and yield another $L(2,1)$ coloring. Formally, the reducibility of an $L(2,1)$ coloring can be defined as follows. Let $C(G)$ be the set of $L(2,1)$-colorings of G. A coloring $f \in C(G)$ is *reducible* if there exists $g \in C(G)$ such that

$g(u) \leq f(u)$ for all $u \in V(G)$ and $g(v) < f(v)$ for some $v \in V(G)$. We denote this reduction from f to g by $g < f$.

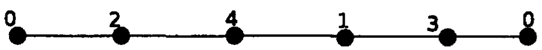
An $L(2,1)$-coloring f is *irreducible* if it is not reducible. A graph G is inh-colorable if there exists an irreducible no-hole coloring of G. Figure 1 provides examples of the above discussed variations of $L(2,1)$ colorings.



(a) An irreducible $L(2,1)$ coloring.

(b) A reducible $L(2,1)$ coloring.

(c) A reducible no-hole $L(2,1)$ coloring.



(d) An irreducible no-hole $L(2,1)$ coloring.

Figure 1: Examples of variations of $L(2,1)$ colorings

Theorem 2. [3] *Let $C(G)$ be the set of all $L(2,1)$-colorings of G. A coloring $f \in C(G)$ is reducible if and only if there exists $g \in C(G)$ with $g < f$ and $g(v) < f(v)$ for only one $v \in V(G)$.*

Let G be an inh-colorable graph. For any inh-coloring f of G let $k = \max f(u)$, $u \in V(G)$. The lower irreducible no-hole span $\lambda_{\text{inh}}(G)$ and the upper irreducible no-hole span $\Lambda_{\text{inh}}(G)$ are the minimum and maximum k respectively, over all inh-colorings f on G. In other words,

$$\lambda_{\text{inh}}(G) = \min_f \{\max f(u) : u \in V(G), f \text{ is an inh-coloring on } G\}$$

$$\Lambda_{\text{inh}}(G) = \max_f \{\max f(u) : u \in V(G), f \text{ is an inh-coloring on } G\}$$

If G is not inh-colorable, then $\lambda_{\text{inh}}(G) = \Lambda_{\text{inh}}(G) = \infty$.

Note that not every graph is inh-colorable. For example, a complete graph is not no-hole colorable and hence not inh-colorable. On the other hand there are graphs with the property that every no-hole coloring is reducible. One such example is shown in Figure 2.

Several classes of graphs have been investigated for their inh-colorability, including trees, paths, cycles [3], unicyclic graphs and hex graphs [8]. One

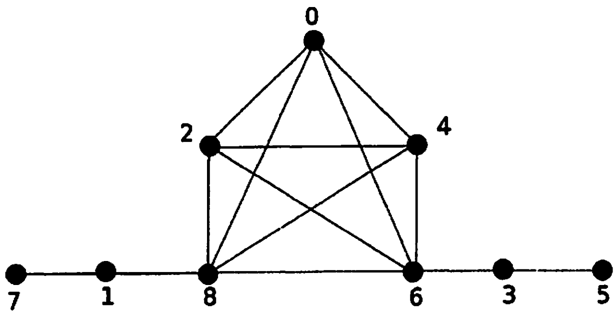
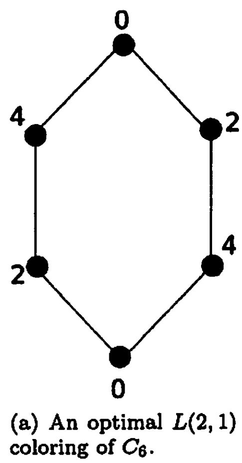
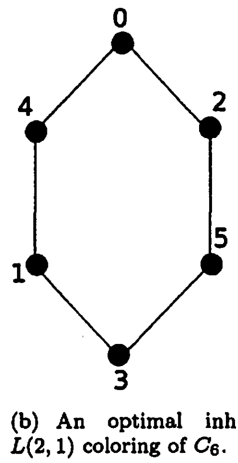


Figure 2: For this graph, every no-hole $L(2,1)$ coloring is reducible

of the interesting and challenging problems is to determine whether for a graph G, $\lambda(G) = \lambda_{\text{inh}}(G)$. It has been shown in [7] that for a tree T, other than a star, $\lambda(T) = \lambda_{\text{inh}}(T)$. Note that not every graph has this property, for example $\lambda(C_6) = 4$ and $\lambda_{\text{inh}}(C_6) = 5$ as shown in Figure 3.



(a) An optimal $L(2,1)$ coloring of C_6.



(b) An optimal inh $L(2,1)$ coloring of C_6.

Figure 3: $4 = \lambda(C_6) < \lambda_{\text{inh}}(C_6) = 5$

In this paper we discover three more classes of graph with the property that $\lambda(G) = \lambda_{\text{inh}}(G)$. In Section 2 we investigate the inh-colorability of bipartite graphs. We provide some sufficient conditions for a bipartite graph to be inh-colorable. In Section 3 we concentrate our efforts on finding the inh span of a special class of bipartite graphs, grid graphs, and show that the span of a grid graph is the same as its inh span.

The study of the inh-colorability of a grid graph is important because many wireless networks have a grid structure. Also, grid graphs are one of the basic Cartesian products, which are defined as follows. Let G and H be two graphs. The *Cartesian product* of G and H, denoted by $G \Box H$, is defined as follows: the vertex set $V(G \Box H) = V(G) \times V(H)$, and two vertices (a, b) and (x, y) are adjacent if and only if either $a = x$ and $by \in E(H)$, or $b = y$ and $ax \in E(G)$. For example, a grid graph is the Cartesian product of two paths. In Section 4 we determine the inh span for the Cartesian product of two complete graphs and the Cartesian product of a complete graph and a path and show that the graphs in these two classes also satisfy the property that $\lambda(G) = \lambda_{\text{inh}}(G)$. We conclude this paper in Section 5 with some open problems.

2 Bipartite Graphs

We begin this section by showing that a complete bipartite graph is not inh-colorable.

Observation 3. *If G is a complete bipartite graph then G is not inh-colorable.*

Proof. Let G be a complete bipartite graph with independent sets S_1 and S_2. We first note that $diam(G) = 2$. Thus, each color class of any $L(2,1)$-coloring of G may contain at most one vertex. Let f be an $L(2,1)$-coloring on G. Let $i = \max(f(u))$ where $u \in S_1$, and let $j = \max(f(v))$ where $v \in S_2$. Now $i \neq j$ since each color class has at most one vertex. Wlog, assume $i < j$. Then the color class $i + 1$ is empty since every vertex in S_2 is adjacent to the vertex labeled i and no vertex in S_1 has a label greater than i. This creates a hole at $i + 1$ since there is a vertex with label larger than $i + 1$, namely j. □

Next we look at a few conditions that are sufficient for a bipartite graph to be inh-colorable.

Theorem 4. *Let G be a bipartite graph with independent sets S_1 and S_2 of cardinalities n and m respectively. Suppose there exist vertices $v_1, v_2 \in S_1$ such that $N(v_1) = S_2$ and $\lfloor \frac{m}{3} \rfloor < |N(v_2)| < m$. Then G is inh-colorable.*

Proof. Let G be a bipartite graph with independent sets S_1 and S_2 of cardinalities n and m respectively where there exist vertices $v_1, v_2 \in S_1$ such that $N(v_1) = S_2$ and $\lfloor \frac{m}{3} \rfloor < |N(v_2)| < m$.

Since $\lfloor \frac{m}{3} \rfloor < |N(v_2)| < m$ there exists an ordering of the vertices of S_2 as $\{u_1, u_2, u_3, \ldots, u_m\}$ where $u_{3i-2}v_2 \in E(G)$ for $i = 1, 2, \ldots, \lfloor \frac{m}{3} \rfloor$, $u_{m-1}v_2 \in E(G)$ and $u_m v_2 \notin E(G)$. Note v_2 may be adjacent to other vertices in S_2.

Define $f(u_i) = i - 1$ for $i = 1, 2, \ldots, m$ and $f(v_2) = m$. Since $N(v_1) = S_2$, $d(u_i, u_j) = 2$ for all $u_i, u_j \in S_2$ and hence no vertex in S_2 can be reduced. Since v_2 is adjacent to a vertex labeled $0, 3, 6, ..., m - (m \mod 3) - 3$, and $m - 2$, its label cannot be reduced. Since v_2 is not adjacent to u_m, f does not violate any $L(2,1)$ constraints.

To finish the labeling f, the remaining vertices in S_1 can be labelled greedily. Since these vertices are labeled greedily the resulting labeling must be irreducible. The labelling f is an $L(2,1)$-coloring, and hence all we need to show is that it contains no holes. The labels $0, 1, 2, \ldots, m-1$ are all used to label S_2. The label m is used to label v_2. Since no vertex in S_1 is adjacent to a vertex labelled m or greater when labelled greedily, the label cannot create a hole. Thus f is an inh-coloring of G. □

Theorem 5. *Let G be a bipartite graph with independent sets S_1 and S_2 where $|S_1| = |S_2|$. If $\lfloor \frac{|S_1|}{2} \rfloor + 1 \leq deg(w) < |S_1|$ for all $w \in V(G)$ then G is inh-colorable.*

Proof. Let G be a bipartite graph with independent sets S_1, S_2 where $|S_1| = |S_2| = m$. Assume $\lfloor \frac{S_1}{2} \rfloor + 1 \leq deg(w) \leq |S_1| - 1$ for all $w \in V(G)$

Since $\lfloor \frac{S_1}{2} \rfloor + 1 \leq |N(u)| \leq |S_1| - 1$ where $u \in S_2$ we can order the vertices in $S_1 = \{v_1, v_2, v_3, ..., v_m\}$ such that $uv_m \notin E(G)$, $uv_{m-1-2i} \in E(G)$ for $i = 0, 1, \ldots, \lfloor \frac{m-2}{2} \rfloor$. Note that u may be adjacent to other v_j.

Let $v_i, v_j \in S_1$. Since $deg(v_i) > \frac{|S_2|}{2}$ and $deg(v_j) > \frac{|S_2|}{2}$, by the pigeon hole principle, v_i and v_j have a common neighbor in S_2. Therefore every vertex in S_1 is at distance two from every other vertex in S_1.

Greedily label the vertices in S_1 as ordered earlier. Since every vertex in S_1 is at distance two from every other vertex in S_1, they must all receive different labels and v_m will receive label $m - 1$. Next greedily label u. Since it is adjacent to v_{m-1-2i} for $i = 0, 1, \ldots, \lfloor \frac{m-2}{2} \rfloor$ and not adjacent v_m, it is labeled m. Finish the labeling by greedily labeling the remaining independent vertices, noting that no vertex in S_2 is adjacent to a vertex labeled m or greater. The coloring is an inh-coloring. □

Corollary 6. *The Crown is inh-colorable.*

Proof. The crown has independent sets S_n and S_m such that $|S_n| = |S_m|$.

It is also the case that $deg(u) = |S_n| - 1$ for all vertices u in the crown. By Theorem 5 the crown is inh-colorable. □

We conclude this section with the following conjecture.

Conjecture 1. *If G is a bipartite graph that is not complete then G is inh-colorable.*

3 Grid Graphs, $P_n \square P_m$

Colorings on grid graphs are well studied because grid graphs serve as graph theoretical models for wireless networks. For examples see [1, 2]. In this section we will determine the inh span of a grid graph.

Consider the grid graph, $P_n \square P_m$, where $n \geq 3$ and $m \geq 3$. Let $v_{i,j}$ denote the vertex in the i^{th} row and j^{th} column.

Let $\{b_n\}$ be the sequence 0, 2, 4, 6, 1, 3, 5, 0, 2, 4, 6, 1, 3, 5, 0, 2, ... Define a coloring f of $P_n \square P_m$ as follows:

- Label the vertices in the first row starting from $v_{1,1}$ using $\{b_n\}$ starting from 0. That is $f(v_{1,1}) = 0, f(v_{1,2}) = 2, \ldots$.
- For $i = 2, 3, \ldots, n$, $f(v_{i,1}) = f(v_{i-1,3})$, and continue the sequence $\{b_n\}$. That is, if $f(v_{i,j}) = b_k$, then $f(v_{i,j+1}) = b_{k+1}$. For example, when $i = 2$, $f(v_{2,1}) = 4, f(v_{2,2}) = 6, f(v_{2,3}) = 1, \ldots$ and when $i = 3$, $f(v_{2,1}) = 1, f(v_{2,2}) = 3, \ldots$.
- If $m \equiv 5 \mod 7$, then change $f(v_{1,m})$ from 1 to 0.
- if $m \equiv 3 \mod 7$, then change $f(v_{2,m})$ from 1 to 0.

An example of the above labeling is shown in Figure 4.

Lemma 7. *Consider $P_n \square P_m$, where $n \geq 3$ and $m \geq 3$ and the coloring f of $P_n \square P_m$ as defined above. For $i = 1, 2, \ldots, n$, if $1 \leq j \leq m-1$ then $|f(v_{i,j}) - f(v_{i,j+1})| \geq 2$, and if $1 \leq j \leq m-2$ then $|f(v_{i,j}) - f(v_{i,j+2})| \geq 1$.*

Proof. Case 1: $m \not\equiv 5 \mod 7$ and $\not\equiv 3 \mod 7$.

Since every row follows the sequence $\{b_n\}$, $|f(v_{i,j}) - f(v_{i,j+1})| = |b_k - b_{k+1}| \geq 2$ and $|f(v_{i,j}) - f(v_{i,j+2})| = |b_k - b_{k+2}| \geq 3 > 1$.

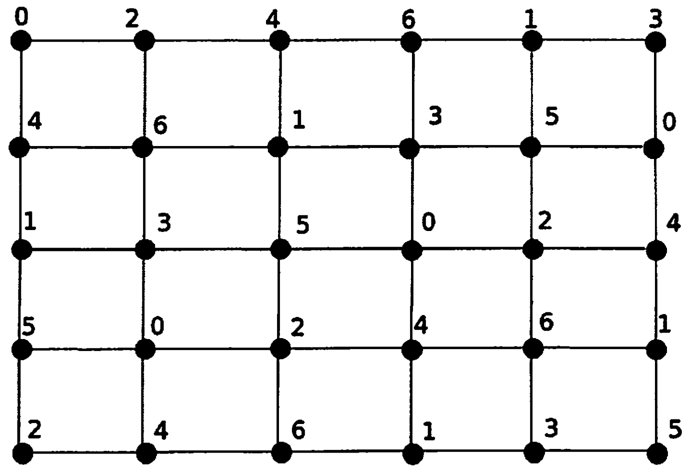


Figure 4: A coloring of $P_5 \square P_6$.

Case 2: $m \equiv 5 \mod 7$ or $m \equiv 3 \mod 7$.

Wlog, assume $m \equiv 5 \mod 7$.

If $i = 1$ and $j = m - 1$, then by definition, $f(v_{i,j}) = 6$ and $f(v_{i,j+1}) = 0$. Also, if $i = 1$ and $j = m - 2$, then by definition of f, $f(v_{i,j}) = 4$ and $f(v_{i,j+2}) = 0$.

However, for other values of i or j, $f(v_{i,j}) = b_k$, $f(v_{i,j+1}) = b_{k+1}$, and if $j \neq m - 1$, then $f(v_{i,j+2}) = b_{k+2}$ where b_k is the k^{th} term of the sequence $\{b_n\}$. Therefore, $|f(v_{i,j}) - f(v_{i,j+1})| = |b_k - b_{k+1}| \geq 2$ and $|f(v_{i,j}) - f(v_{i,j+2})| = |b_k - b_{k+2}| \geq 3 > 1$. Hence the lemma. □

Lemma 8. *Consider $P_n \square P_m$, where $n \geq 3$ and $m \geq 3$ and the coloring f of $P_n \square P_m$ as defined above. For $j = 1, 2, \ldots, m$, if $1 \leq i \leq n - 1$ then $|f(v_{i,j}) - f(v_{i+1,j})| \geq 2$, and if $1 \leq i \leq n-2$ then $|f(v_{i,j}) - f(v_{i+2,j})| \geq 1$.*

Proof. Case 1: $m \not\equiv 5 \mod 7$ and $\not\equiv 3 \mod 7$.

Since for every row $i = 2, 3, \ldots, n$, $f(v_{i,1}) = f(v_{i-1,3})$ and the sequence $\{b_n\}$ is followed, we have $f(v_{i,j}) = b_k$ and $f(v_{i+1,j}) = b_{k+2}$ and $f(v_{i+2,j}) = b_{k+4}$. Thus we have $|f(v_{i,j}) - f(v_{i+1,j})| = |b_k - b_{k+2}| \geq 3$ and $|f(v_{i,j}) - f(v_{i+2,j})| = |b_k - b_{k+4}| \geq 1$.

Case 2: $m \equiv 5 \mod 7$ or $m \equiv 3 \mod 7$

Wlog, assume $m \equiv 5 \mod 7$.

Suppose $j \neq m$. Then as in the previous case, since for every row $i = 2, 3, \ldots, n$, $f(v_{i,1}) = f(v_{i-1,3})$ and the sequence $\{b_n\}$ is followed, we have $|f(v_{i,j}) - f(v_{i+1,j})| = |b_k - b_{k+2}| \geq 3$ and $|f(v_{i,j}) - f(v_{i+2,j})| = |b_k - b_{k+4}| \geq 1$.

Suppose $j = m$. If $i = 1$, then $f(v_{i,j}) = f(v_{1,m}) = 0$, $f(v_{i+1,j}) = f(v_{2,m}) = 5$ and $f(v_{i+2,j}) = f(v_{3,m}) = 2$. Therefore, $|f(v_{i,j}) - f(v_{i+1,j})| = 5$, and $|f(v_{i,j}) - f(v_{i+2,j})| = 2$. If $i \geq 2$, then $f(v_{i,1}) = f(v_{i-1,3})$ and the sequence $\{b_n\}$ is followed, we have $|f(v_{i,j}) - f(v_{i+1,j})| = |b_k - b_{k+2}| \geq 3$ and $|f(v_{i,j}) - f(v_{i+2,j})| = |b_k - b_{k+4}| \geq 1$. Hence the lemma. □

Lemma 9. *Consider $P_n \square P_m$, where $n \geq 3$ and $m \geq 3$ and the coloring f of $P_n \square P_m$ as defined above. For $1 \leq i \leq n-1$, if $1 \leq j \leq m-1$, then $|f(v_{i,j}) - f(v_{i+1,j+1})| \geq 1$, and if $2 \leq j \leq m$, then $|f(v_{i,j}) - f(v_{i+1,j-1})| \geq 1$.*

Proof. Case 1: $m \not\equiv 5 \mod 7$ and $\not\equiv 3 \mod 7$.

Since for every row $i = 1, 2, \ldots, n-1$, $f(v_{i+1,1}) = f(v_{i,3})$ and the sequence $\{b_n\}$ is followed, we have $f(v_{i,j}) = b_k$, $f(v_{i+1,j+1}) = b_{k+3}$ and $f(v_{i+1,j-1}) = b_{k+1}$. Therefore, by the definition of $\{b_n\}$, we have $|f(v_{i,j}) - f(v_{i+1,j+1})| = |b_k - b_{k+3}| \geq 1$ and $|f(v_{i,j}) - f(v_{i+1,j-1})| = |b_k - b_{k+1}| \geq 2$.

Case 2: $m \equiv 5 \mod 7$ or $m \equiv 3 \mod 7$

Wlog, assume $m \equiv 5 \mod 7$.

Suppose $2 \leq i \leq n-1$ or $1 \leq j \leq m-1$. Since for every row $i = 1, 2, \ldots, n-1$, $f(v_{i+1,1}) = f(v_{i,3})$ and the sequence $\{b_n\}$ is followed, we have $f(v_{i,j}) = b_k$, $f(v_{i+1,j+1}) = b_{k+3}$ and $f(v_{i+1,j-1}) = b_{k+1}$ and thus $|f(v_{i,j}) - f(v_{i+1,j+1})| \geq 1$ and $|f(v_{i,j}) - f(v_{i+1,j-1})| \geq 2$.

If $i = 1$ and $j = m$, then $|f(v_{i,j}) - f(v_{1+1,j-1})| = |f(v_{1,m}) - f(v_{2,m-1})| = |0 - 3| = 3$. Hence the lemma. □

Theorem 10. *Let $n \geq 3$ and $m \geq 3$. Then $\lambda_{inh}(P_n \square P_m) \leq 6$.*

Proof. Consider the coloring f as defined above for a grid graph $P_n \square P_m$, where $n \geq 3$ and $m \geq 3$. We will show that f is an inh $L(2,1)$ coloring.

By Lemmas 7, 8 and 9, we can see that f is an $L(2,1)$ coloring. Also, since f uses all labels from 0 to 6, f is a no-hole $L(2,1)$-coloring. We will now show that f is irreducible.

Consider $f(v_{i,j})$. We will show that $f(v_{i,j})$ cannot be reduced to label $a < f(v_{i,j})$ because of $f(v_{p,q})$ where either $1 \le p \le 3$ or $p \le i$, and either $1 \le q \le 3$ or $q \le j$.

Case 1: $f(v_{i,j}) = 1$.

If $i \ge 3$, then $f(v_{i-2,j}) = 0$, by definition. If $i < 3$, then $f(v_{i+1,j+1}) = 0$ by the definition of f. Therefore $f(v_{i,j})$ cannot be reduced to a smaller label.

Case 2: $f(v_{i,j}) = 2$.

$f(v_{i,j})$ cannot be reduced, because by the definition of f, $f(v_{i,j-1}) = 0$ if $j > 1$, and if $j = 1$, then $f(v_{i-2,j}) = 1$ and $f(v_{i-1,j+1}) = 0$. Therefore $f(v_{i,j})$ cannot be reduced to a smaller label.

Case 3: $f(v_{i,j}) = 3$.

$f(v_{i,j})$ cannot be reduced, because by the definition of f, $f(v_{i,j-1}) = 1$ if $j > 1$, and if $j = 1$, then $f(v_{i-2,j}) = 2$, $f(v_{i-1,j+1}) = 1$ and $f(v_{i,j+2}) = 0$. Therefore $f(v_{i,j})$ cannot be reduced to a smaller label.

Case 4: $f(v_{i,j}) = 4$.

If $i = 1$, then $f(v_{i,j-1}) = 2$ and $f(v_{i,j-2}) = 0$. If $i > 1$ and $j > 1$, then by definition, $f(v_{i-1,j}) = 0$, $f(v_{i,j-1}) = 2$. If $j = 1$ and $i = 2$, then $f(v_{i-1,j}) = 0$, $f(v_{i-1,j+1}) = 2$, and $f(v_{i+1,j+1}) = 3$. And finally if $j = 1$ and $i > 2$, we have $f(v_{i-1,j}) = 0$, $f(v_{i-1,j+1}) = 2$, and $f(v_{i-2,j}) = 3$.

Therefore, $f(v_{i,j})$ cannot be reduced to a smaller label.

Case 5: $f(v_{i,j}) = 5$.

If $i = 1$, then $f(v_{i,j-1}) = 3$, $f(v_{i,j-2}) = 1$ and $f(v_{i+1,j-1}) = 0$. If $j = 1$, then $f(v_{i-1,j}) = 1$, $f(v_{i-1,j+1}) = 3$, and $f(v_{i-2,j}) = 4$. If $i > 1$ and $j > 1$, then $f(v_{i,j-1}) = 3$, $f(v_{i-1,j}) = 1$ if $i > 3$ or $j \neq m$ and 0 otherwise.

Therefore, $f(v_{i,j})$ cannot be reduced to a smaller label.

Case 6: $f(v_{i,j}) = 6$.

If $i = 1$, then $f(v_{i,j-1}) = 4$, $f(v_{i+1,j}) = 3$, $f(v_{i+1,j-1}) = 1$ and $f(v_{i+2,j}) = 0$. If $j = 1$, then $f(v_{i,j+1}) = 1$, $f(v_{i-1,j}) = 2$, $f(v_{i-1,j+1}) = 4$ and $f(v_{i-2,j}) = 5$. If $i > 1$ and $j > 1$, then by definition, $f(v_{i,j-1}) = 4$, $f(v_{i-1,j}) = 2$ and $f(v_{i-1,j-1}) = 0$.

Therefore, $f(v_{i,j})$ cannot be reduced to a smaller label.

Thus in all cases $f(v_{i,j})$ cannot be reduced to a smaller label because of $f(v_{p,q})$ such that either $1 \le p \le 3$ or $p \le i$, and either $1 \le q \le 3$ or $q \le j$.

Therefore, f is irreducible, and hence f is an inh $L(2,1)$ coloring.

Since the largest label under f is 6, we have $\lambda_{\text{inh}}(P_n \square P_m) \leq 6$. □

Theorem 11. *Let* $n \geq 4$ *and* $m \geq 4$. $\lambda_{inh}(P_n \square P_m) = 6$.

Proof. Let $n \geq 4$ and $m \geq 4$. In [10] the authors have shown that $\lambda(P_n \square P_m) = 6$ if $n \geq 4$ and $m \geq 4$. However, we know that $\lambda(G) \leq \lambda_{\text{inh}}(G)$ for any graph G and therefore we have

$$\lambda_{\text{inh}}(P_n \square P_m) \geq \lambda(P_n \square P_m) = 6. \quad (1)$$

Hence, from (1) and Theorem 10 we get $\lambda_{\text{inh}}(P_n \square P_m) = 6$. □

Corollary 12. $\lambda_{inh}(P_n \square P_m) = \lambda(P_n \square P_m)$, *if* $n > 3$ *and* $m > 3$.

4 More Cartesian products

In this section we will investigate the inh-colorability of two more Cartesian products, the Cartesian product of two complete graphs, also known as the rook's graph, and the Cartesian product of a complete graph and a path, and calculate their inh span.

First we will consider $K_n \square K_m$, where $n > 2$ and $m > 2$.

Theorem 13. $\lambda_{inh}(K_n \square K_m) = mn - 1$.

Proof. Consider the function $f : V(K_n \square K_m) \longrightarrow \{0, 1, \ldots, mn-1\}$ defined by $f(v_{i,j}) = ((i-1) + (j-i)n) \mod mn$. First we will show that the function f is a bijection.

Let $f(v_{i,j}) = f(v_{a,b})$ where $1 \leq i, a \leq n$ and $1 \leq j, b \leq m$.

That is,

$$\begin{aligned} ((i-1) + (j-i)n) \mod mn &= ((a-1) + (b-a)n) \mod mn \\ \Rightarrow (i-1) \mod n &= (a-1) \mod n \\ \Rightarrow i &= a, \text{ because } 1 \leq i, a \leq n. \end{aligned}$$

Since $i = a$ and $f(v_{i,j}) = f(v_{a,b})$, we have $(j-i) \mod m = (b-i) \mod m$. However, $1 \leq j, b \leq m$ and therefore $j = b$. Thus we have shown that f is an injection.

However $f : V(K_n \Box K_m) \rightarrow \{0, 1, \ldots, mn-1\}$ and $|V(K_n \Box K_m)| = |\{0, 1, \ldots, mn-1\}|$. Also, we have shown that f is an injection, and hence f is a bijection.

Since f is a bijection, f is a no-hole coloring. Now we will show that f is an $L(2,1)$ coloring.

Let x and y be two adjacent vertices of $K_n \Box K_m$.

Since $f(v_{i,j}) = ((i-1)+(j-i)n) \mod mn$, if x and y are in the same row, then $|f(x) - f(y)|$ is a multiple of n. However, since f is a bijection $|f(x) - f(y)| \neq 0$. Therefore, if $x = v_{i,j}$ and $y = v_{i,k}$ (i.e, x and y are in the same row), then $|f(v_{i,j}) - f(v_{i,k})| \geq n \geq 3$.

Similarly, since $f(v_{i,j}) = ((i-1)+(j-i)n) \mod mn = (i(1-n)+jn-1) \mod mn$, if x and y are in the same column $|f(x) - f(y)| \geq n-1 \geq 2$.

Also, since f is a bijection, $f(x) \neq f(y)$ for all $x, y \in V(K_n \Box K_m)$. Therefore, we have shown that f is an $L(2,1)$ coloring.

Since $d(x,y) \leq 2$ for all $x, y \in V(K_n \Box K_m)$, $f(x)$ cannot be reduced to a smaller label and still be an $L(2,1)$ coloring.

Therefore, f is an inh $L(2,1)$ coloring. Since the largest color used in f is $mn-1$,

$$\lambda_{\text{inh}}(K_n \Box K_m) \leq mn-1. \tag{2}$$

However, since $d(x,y) \leq 2$ for all $x, y \in V(K_n \Box K_m)$, any $L(2,1)$ coloring of $K_n \Box K_m$ must use mn colors. Wlog, assume that the smallest label used in an $L(2,1)$ coloring is 0. Thus we have,

$$\lambda_{\text{inh}}(K_n \Box K_m) \geq \lambda(K_n \Box K_m) \geq mn-1. \tag{3}$$

Therefore, from (2) and (3), we have $\lambda_{\text{inh}}(K_n \Box K_m) = mn-1$. □

An example of an optimal inh $L(2,1)$ coloring of $K_3 \Box K_4$ is shown in Figure 5.

Corollary 14. *$\lambda_{inh}(K_n \Box K_m) = \lambda(K_n \Box K_m)$, if $n > 2$ and $m > 2$.*

Next we will consider the Cartesian product $K_n \Box P_m$ where $n > 3$ and $m > 1$.

Theorem 15. *Let $n > 3$ and $m > 1$. Then $\lambda_{inh}(K_n \Box P_m) = 2n-1$.*

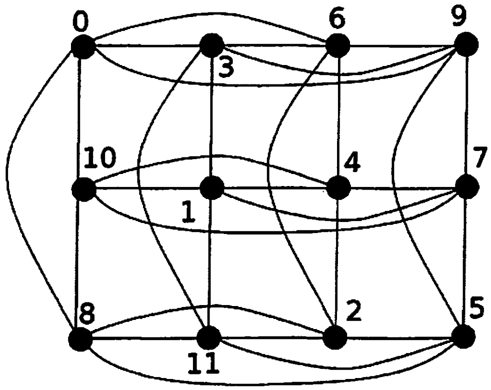


Figure 5: An inh $L(2,1)$ coloring of $K_3 \square K_4$.

Proof. Let $v_0, v_1, \ldots, v_{n-1}$ be the vertices of K_n. Assume $v_{i,j}$ denotes the i^{th} vertex of K_n (i.e, v_i) in layer j of $K_n \square P_m$.

Define a coloring f as follows:

- $f(v_{0,1}) = 0$, $f(v_{1,1}) = 2$, $f(v_{2,1}) = 4$, ..., $f(v_{n-1,1}) = 2(n-1)$.
- $f(v_{0,2}) = 3$, $f(v_{1,2}) = 5$, $f(v_{2,2}) = 7$, ..., $f(v_{n-2,2}) = 2(n-2)+3$, $f(v_{n-1,2}) = 1$.
- $f(v_{i,j}) = f(v_{k,j-2})$, where $k = (i-1) \mod n$, $0 \leq i \leq n-1$ and $3 \leq j \leq m$.

Note that for layer $j > 2$, to get the coloring we rotate the coloring of layer $j-2$ clockwise one position. See Figure 6 for an example.

Claim: f is an $L(2,1)$ coloring.

Let x and y be two adjacent vertices in $K_n \square P_m$. If x and y are in the same layer, then by definition of f, $|f(x) - f(y)| \geq 2$. On the other hand, let x and y be in adjacent layers, say $x = v_{i,j}$ and $y = v_{i,j+1}$. If $j = 1$, then by definition, $|f(x) - f(y)| \geq 3$. Similarly, since the labeling of layer 3 is the labeling of layer 1 rotated clockwise one position and $n > 3$, if $j = 2$ then $|f(x) - f(y)| \geq 3$. Also, since we rotate the labels clockwise one position, if $j > 2$, then for $k = i - \lfloor \frac{j-1}{2} \rfloor \mod n$

$$f(x) = f(v_{i,j}) = \begin{cases} f(v_{k,1}), \text{ if } j \text{ is odd} \\ f(v_{k,2}), \text{ if } j \text{ is even} \end{cases}$$

and

$$f(y) = f(v_{i,j+1}) = \begin{cases} f(v_{k,2}), \text{ if } j \text{ is odd} \\ f(v_{k,3}), \text{ if } j \text{ is even.} \end{cases}$$

Therefore, $|f(x) - f(y)| \geq 3$.

Now suppose x and y are at a distance two. If x and y are in adjacent layers, then by definition, $f(x) \neq f(y)$. On the other hand, if $x = v_{i,j}$ and $y = v_{i,j+2}$, (i.e, x is in layer j and y is in layer $j+2$), then again by definition $f(x) \neq f(y)$ (because the coloring of layer $j+2$ is obtained by rotating the colors of layer j one position clockwise).

Thus, f is an $L(2,1)$ coloring, and hence the claim.

We will now show that f is an inh $L(2,1)$ coloring. By definition, f uses all colors from 0 to $2n-1$ in the first two layers, and since layer $j > 2$ is colored by rotating the colors of layer $j-2$ clockwise one position, the largest label used under f is $2n-1$. Thus f is a no-hole coloring. Also, any two adjacent layers will have labels from 0 to $2n-1$, and $d(x,y) \leq 2$ if x and y are in adjacent layers. So for any label $f(x) > 0$, $f(x)$ cannot be reduced to a smaller label and therefore f is irreducible.

Therefore, f is an inh $L(2,1)$ coloring of $K_n \square P_m$, and thus we have,

$$\lambda_{\text{inh}}(K_n \square P_m) \leq |f| = 2n - 1. \tag{4}$$

In addition, any two vertices x and y in $K_n \square P_m$ that are in the adjacent layers have $d(x,y) \leq 2$. Since there are at least two layers (because $m \geq 2$), any $L(2,1)$ coloring of $K_n \square P_m$ must use at least $2n$ colors. This implies,

$$\lambda_{\text{inh}}(K_n \square P_m) \geq \lambda(K_n \square P_m) \geq 2n - 1. \tag{5}$$

Therefore, from (4) and (5) we get $\lambda_{inh}(K_n \square P_m) = 2n - 1$ if $n > 3$ and $m > 1$. □

An example of an optimal inh $L(2,1)$ coloring of $K_5 \square P_4$ is given is Figure 6.

Corollary 16. *$\lambda_{inh}(K_n \square P_m) = \lambda(K_n \square P_m)$, if $n > 3$ and $m > 1$.*

5 Conclusion

In this paper we studied the inh-colorability of certain classes of graphs, including bipartite graphs and certain Cartesian products. We determined

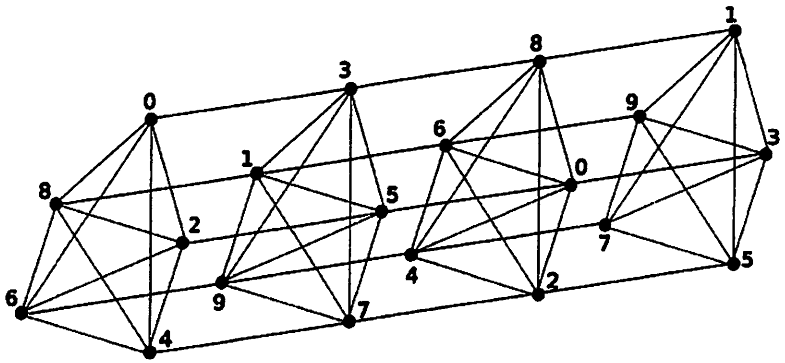


Figure 6: An optimal inh $L(2,1)$ coloring of $K_5 \square P_4$.

the inh span of the following graphs: grid graphs, the Cartesian product of two complete graphs and the Cartesian product of a complete graph and a path. We have also established that the above three classes of graphs have the property that their inh spans are the same as their spans. We believe that these results will help us in answering questions about the inh-colorability of bipartite graphs and Cartesian products.

There are still some more questions that are worth investigating related to these topics. First, are all bipartite graphs, except a complete bipartite graph, inh-colorable? We believe that the answer is yes, and posed this as a conjecture. Also, what other classes of graphs have the property that $\lambda_{\text{inh}}(G) = \lambda(G)$? There are different kinds of grids one could consider while modeling wireless networks and other applications. One example is triangular grids. $P_m \square P_n$ is called the square grid. What can we say about the inh-colorability of grids that are not square grids? In general, can we characterize graphs that are inh-colorable?

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