On Balance Index Sets of Generalized Wheels

Man C. Kong
Dept. of EE & CS
University of Kansas
Lawrence, KS 66045, USA

Sin-Min Lee
Dept. of Comp. Sci.
San Jose State Univ.
San Jose, CA 95192, USA

Herbert A. Evans
Dept. of Comp. Sci.
San Jose State Univ.
San Jose, CA 95192, USA

Harris Kwong
Dept. of Math. Sci.
SUNY at Fredonia
Fredonia, NY 14063, USA

Abstract

A vertex labeling $f : V \to \{0,1\}$ of the simple graph $G = (V, E)$ induces a partial edge labeling $f^* : E \to \{0,1\}$ defined by $f^*(uv) = f(u)$ if and only if $f(u) = f(v)$. Let $v(i)$ and $e(i)$ be the number of vertices and edges, respectively, that are labeled i, and define the balance index set of G as $\{|e(0) - e(1)| : |v(0) - v(1)| \leq 1\}$. In this paper, we determine the balance index sets of generalized wheels, which are the Zykov sum of a cycle with a null graph.

1 Introduction

Lee, Liu and Tan [11] considered a new labeling problem in graph theory. Given any vertex labeling $f : V \to \{0,1\}$ of a simple graph $G = (V, E)$, define a partial edge labeling f^* of G according to

$$f^*(uv) = \begin{cases} 0 & \text{if } f(u) = f(v) = 0, \\ 1 & \text{if } f(u) = f(v) = 1. \end{cases}$$

Note that the edge uv is unlabeled if $f(u) \neq f(v)$.

Denote by $v_f(0)$ and $v_f(1)$ the number of vertices of G that are labeled 0 and 1, respectively, under the mapping f. Analogously, let $e_f(0)$ and $e_f(1)$ denote the number of edges of G that are labeled 0 and 1, respectively, by the induced partial function f^*. When the context is clear, we will simply write $v(0)$, $v(1)$, $e(0)$, and $e(1)$ without any subscript.

JCMCC **78** (2011), pp. 65-74

Definition 1.1. A vertex labeling f of a graph G is said to be ***friendly*** if $|v_f(0) - v_f(1)| \le 1$, and ***balanced*** if f is friendly *and* $|e_f(0) - e_f(1)| \le 1$.

Call a graph ***balanced*** if it admits a balanced labeling. See [2, 3, 15] for further results in balanced graphs. It is clear that not all graphs are balanced. Lee, Lee and Ng [9] introduced the following concept as an extension of their study of balanced graphs.

Definition 1.2. The ***balance index set*** of the graph G is defined as

$$\mathrm{BI}(G) = \{|e_f(0) - e_f(1)| : \text{the vertex labeling } f \text{ is friendly}\}.$$

Example 1. It is not difficult to verify that the balance index set of the graph G displayed in Figure 1 is $\{0, 1, 2\}$. □

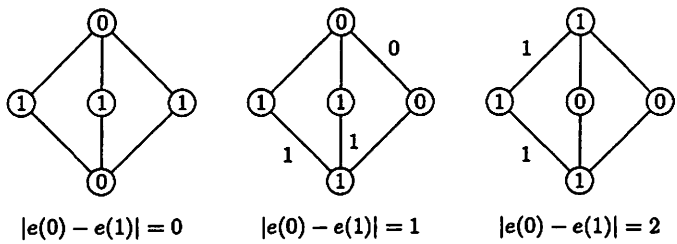

Figure 1: The friendly labelings of a graph G with $\mathrm{BI}(G) = .\{0, 1, 2\}$.

In general, it is a difficult task to determine the balance index set of a given graph. Balance index sets of special families of graphs with relatively simple structures had been found [2, 4, 9, 14, 17]. Examples include

$$\mathrm{BI}(\mathrm{St}(n)) = \begin{cases} \{k\} & \text{if } n = 2k+1, \\ \{k-1, k\} & \text{if } n = 2k, \end{cases}$$

and

$$\mathrm{BI}(C_n(t)) = \begin{cases} \{0, 1\} & \text{if } n \text{ is even}, \\ \{0, 1, 2\} & \text{if } n \text{ is odd}, \end{cases}$$

where $\mathrm{St}(n)$ is the star with n pendant vertices, and $C_n(t)$ denotes an n-cycle with a chord connecting two nonadjacent vertices at distance $t - 1$ apart on the cycle. Another interesting example can be find in [7]:

Theorem 1.1 *If G is a k-regular graph of order p, then*

$$BI(G) = \begin{cases} \{0\} & \text{if } p \text{ is even}, \\ \{k/2\} & \text{if } p \text{ is odd}. \end{cases}$$

Graphs with more complicated structure such as permutation graphs, Halin graphs, chain-sum of cycles, and those formed by the amalgamation of complete graphs, stars, and generalized theta graphs, and L-products with cycles and complete graphs were studied in [1, 5, 6, 8, 10, 12, 13].

The Zykov sum of the n-cycle C_n with the null graph N_m is called a ***generalized wheel***, and is denoted $\mathrm{GW}(n,m)$. It is a wheel with m centers (or hubs) $c_1, c_2, \ldots, c_m$, each of which is connected to the n vertices on the rim. Therefore it has mn spokes, and n edges on the rim. When $m = 1$, we have the usual wheel W_n. In this paper, we determine the balance index set of $\mathrm{GW}(n,m)$ for all integers $n \geq 3$ and $m \geq 1$.

2 The BI Sets of Wheels

Most of the existing results on balance index sets are derived via an ad hoc approach, which relies on the specific structures of the graphs being studied. It was remarked in [4] that $\mathrm{BI}(G)$ depends on the degree sequence of G. This idea was further explored in [16], in which an algebraic approach was proposed. Following the same spirit, it is easy to prove the next lemma.

Lemma 2.1 *If e^* denotes the number of unlabeled edges in the induced edge labeling of a graph, then for $i = 0, 1$,*

$$2e(i) + e^* = \sum_{f(v)=i} \deg(v),$$

and

$$2|E(G)| = \sum_{v \in V(G)} \deg(v) = \sum_{f(v)=0} \deg(v) + \sum_{f(v)=1} \deg(v).$$

This lemma immediately leads to our key theorem.

Theorem 2.2 *For any friendly vertex labeling, the balance index is*

$$e(0) - e(1) = \frac{1}{2}\left(\sum_{f(v)=0} \deg(v) - \sum_{f(v)=1} \deg(v)\right).$$

In particular, when G is k-regular, we obtain Theorem 1.1. The wheel graph W_n is bi-regular: all the vertices on the rim have degree 3, but the center has degree n.

Theorem 2.3 ([10]) *The balance index set of $GW(n,1)$ is $\{\frac{n-3}{2}\}$ for all odd integers $n \geq 3$.*

Proof. If n is odd, the wheel graph has $n+1$ vertices, thus $v(0) = v(1) = \frac{n+1}{2}$. Notice that if f is a friendly labeling, by changing the labels of the vertices from 0 to 1, and 1 to 0, the new vertex labeling is still friendly, but $|e(0) - e(1)|$ remains unchanged. Hence we may assume the center of the wheel is an 0-vertex. Then the numbers of 0- and 1-vertices on the rim are $\frac{n-1}{2}$ and $\frac{n+1}{2}$ respectively. It follows from Theorem 2.2 that

$$e(0) - e(1) = \frac{1}{2}\left[n + \frac{3(n-1)}{2} - \frac{3(n+1)}{2}\right] = \frac{n-3}{2}.$$

This completes the proof. □

Theorem 2.4 *The balance index set of* $GW(n,1)$ *is* $\{\frac{n}{2}, |\frac{n}{2} - 3|\}$ *for all even integers* $n \geq 4$.

Proof. We have two possibilities. In the first case, $v(0) - v(1) = -1$, then the number of 0- and 1-vertices on the rim are $\frac{n}{2} - 1$ and $\frac{n}{2} + 1$ respectively. In the second case, $v(0) - v(1) = 1$, then the numbers of 0- and 1-vertices on the rim are both $\frac{n}{2}$. The result follows from Theorem 2.2. □

Example 2. It suffices to describe the vertex labeling on the rim. They are listed below along with their respective BI sets. □

n	labelings of C_n	BI(GW(n, 2))
3	011	{0}
5	00111	{1}
7	0001111	{2}
4	0111, 0011	{1, 2}
6	001111, 000111	{0, 3}
8	00011111, 00001111	{1, 4}

3 The Cases of $2 \leq m \leq 5$

For $m \geq 2$, how the centers are labeled does make a difference. hence we need to analyze the labelings that we could assign to these m centers. Nevertheless, since interchanging 0-vertices with 1-vertices in a friendly labeling does not alter the value of $|e(0) - e(1)|$, we may assume $v(0) \leq v(1)$.

Theorem 3.1 *For* $n \geq 3$,

$$BI(GW(n,2)) = \begin{cases} \{2, |n-2|, |n-6|\} & \text{if } n \text{ is odd,} \\ \{0, n-4\} & \text{if } n \text{ is even.} \end{cases}$$

Proof. If $n = 2k+1$, where $k \geq 1$, the generalized wheel $GW(n,2)$ has $2k+3$ vertices. We may assume $v(0) = k+1$, and $v(1) = k+2$. The vertices on the rim are of degree 4, and the degrees of the two centers c_1 and c_2 are $2k+1$. We need to analyze three cases, depending on the labels of c_1 and c_2. If both are labeled 0, then, on the rim, $k-1$ vertices are labeled 0, and $k+2$ are are labeled 1. According to Theorem 2.2,

$$e(0) - e(1) = \frac{1}{2}\left[2(2k+1) + 4(k-1) - 4(k+2)\right] = 2k-5.$$

The other two cases are handled in a similar fashion. The results are summarized in the following table, in which $n_k(i)$ denotes the number of vertices of degree k on the rim that are labeled i.

$f(c_1)$	$f(c_2)$	$n_4(0)$	$n_4(1)$	$e(0)-e(1)$
0	0	$k-1$	$k+2$	$2k-5$
0	1	k	$k+1$	-2
1	1	$k+1$	k	$1-2k$

The result for odd n follows immediately. The same argument gives the result for even n. □

Theorem 3.2 *For* $n \geq 3$,

$$BI(GW(n,3)) = \begin{cases} \{|k-2|, |3k-6|\} & \text{if } n = 2k+1, \\ \{|k-5|, k, |3k-10|, |3k-5|\} & \text{if } n = 2k. \end{cases}$$

Proof. If $n = 2k+1$, where $k \geq 1$, the generalized wheel $GW(n,3)$ has $2k+4$ vertices, hence $v(0) = v(1) = k+2$. The vertices on the rim are of degree 5, and the degrees of the three centers c_1, c_2 and c_3 are $2k+1$. We need to analyze four cases of values of $f(c_i)$. For example, if each c_i is labeled 0, then $k-1$ vertices on the rim are labeled 0, and $k+2$ are labeled 1. Thus

$$e(0) - e(1) = \frac{1}{2}\left[3(2k+1) + 5(k-1) - 5(k+2)\right] = 3k-6.$$

Results from all four cases are summarized below.

$f(c_1)$	$f(c_2)$	$f(c_3)$	$n_5(0)$	$n_5(1)$	$e(0)-e(1)$
0	0	0	$k-1$	$k+2$	$3k-6$
0	0	1	k	$k+1$	$k-2$
0	1	1	$k+1$	k	$2-k$
1	1	1	$k+2$	$k-1$	$6-3k$

Hence the balance index set is $\{|k-2|, |3k-6|\}$ if $n = 2k+1$. The result for $n = 2k$ is obtained in a similar fashion. □

Theorem 3.3 *While $BI(GW(3,4)) = \{0,3,6\}$, we find, for $n \geq 4$,*

$$BI(GW(n,4)) = \begin{cases} \{3, |n-9|, |n-3|, |2n-15|, |2n-9|\} & \text{if } n \text{ is odd,} \\ \{0, |n-6|, |2n-12|\} & \text{if } n \text{ is even.} \end{cases}$$

Proof. The vertices on the rim are of degree 6, and the degrees of the four centers are n. We need to analyze the values of $f(c_i)$. For $n = 3$, we may assume $v(0) = 3$ and $v(1) = 4$. There are only 4 cases:

$f(c_1)$	$f(c_2)$	$f(c_3)$	$f(c_4)$	$n_6(0)$	$n_6(1)$	$e(0) - e(1)$
0	0	0	1	0	3	−6
0	0	1	1	1	2	−3
0	1	1	1	2	1	0
1	1	1	1	3	0	3

Hence $\text{BI}(\text{GW}(3,4)) = \{0,3,6\}$. If $n = 2k+1$, where $k \geq 2$, we may assume $v(0) = k+2$, and $v(1) = k+3$. The following results lead to the balance index set for odd $n \geq 5$.

$f(c_1)$	$f(c_2)$	$f(c_3)$	$f(c_4)$	$n_6(0)$	$n_6(1)$	$e(0) - e(1)$
0	0	0	0	$k-2$	$k+3$	$2n-15$
0	0	0	1	$k-1$	$k+2$	$n-9$
0	0	1	1	k	$k+1$	−3
0	1	1	1	$k+1$	k	$3-n$
1	1	1	1	$k+2$	$k-1$	$9-2n$

The argument for even n is similar, and is omitted here. □

Theorem 3.4 *$BI(GW(3,5)) = \{2,6\}$, $BI(GW(4,5)) = \{1,2,4,5,8\}$, and for $n \geq 5$, the balance index set of $GW(n,5)$ is*

$$\begin{cases} \{|k-3|, 3|k-3|, 5|k-3|\} & \text{if } n = 2k+1, \\ \{|k-7|, k, |3k-14|, |3k-7|, |5k-21|, |5k-14|\} & \text{if } n = 2k. \end{cases}$$

Proof. In $\text{GW}(n,5)$, the vertices on the rim are of degree 7, the centers are of degree n. We first consider odd n. For $n = 3$, since $\text{GW}(3,5)$ has 8 vertices, we need $v(0) = v(1) = 4$. The four different combinations of the labeling of the centers are tabulated below.

$f(c_1)$	$f(c_2)$	$f(c_3)$	$f(c_4)$	$f(c_5)$	$n_7(0)$	$n_7(1)$	$e(0) - e(1)$
0	0	0	0	1	0	3	−6
0	0	0	1	1	1	2	−2
0	0	1	1	1	2	1	2
0	1	1	1	1	3	0	6

In general, for $n = 2k+1$, where $k \geq 2$, we need $v(0) = v(1) = k+3$. We summarize the various combinations of $f(c_i)$ in the next table.

$f(c_1)$	$f(c_2)$	$f(c_3)$	$f(c_4)$	$f(c_5)$	$n_7(0)$	$n_7(1)$	$e(0)-e(1)$
0	0	0	0	0	$k-2$	$k+3$	$5(k-3)$
0	0	0	0	1	$k-1$	$k+2$	$3(k-3)$
0	0	0	1	1	k	$k+1$	$k-3$
0	0	1	1	1	$k+1$	k	$3-k$
0	1	1	1	1	$k+2$	$k-1$	$3(3-k)$
1	1	1	1	1	$k+3$	$k-2$	$5(3-k)$

This completes the proof of the case of odd n.

When $n = 4$, since GW(4,5) has 9 vertices, we may assume $v(0) = 4$, and $v(1) = 5$. The five different combinations of the vertex labels for c_i are:

$f(c_1)$	$f(c_2)$	$f(c_3)$	$f(c_4)$	$f(c_5)$	$n_7(0)$	$n_7(1)$	$e(0)-e(1)$
0	0	0	0	1	0	4	−8
0	0	0	1	1	1	3	−5
0	0	1	1	1	2	2	−2
0	1	1	1	1	3	1	1
1	1	1	1	1	4	0	4

In general, when $n = 2k$, where $k \geq 3$, the graph GW$(2k, 5)$ has $2k+5$ vertices, we may assume $v(0) = k+2$, and $v(1) = k+3$. From the data depicted below

$f(c_1)$	$f(c_2)$	$f(c_3)$	$f(c_4)$	$f(c_5)$	$n_7(0)$	$n_7(1)$	$e(0)-e(1)$
0	0	0	0	0	$k-3$	$k+3$	$5k-21$
0	0	0	0	1	$k-2$	$k+2$	$3k-14$
0	0	0	1	1	$k-1$	$k+1$	$k-7$
0	0	1	1	1	k	k	$-k$
0	1	1	1	1	$k+1$	$k-1$	$7-3k$
1	1	1	1	1	$k+2$	$k-2$	$14-5k$

we obtain the desired balance index set. □

Example 3. Theorem 3.4 asserts that BI(GW(5,5)) = BI(GW(9,5)) = $\{1,3,5\}$, and, interestingly, BI(GW(7,5)) = $\{0\}$. In fact, the last result is not surprising at all. It agrees with Theorem 1.1 because GW(7,5) is a regular graph. □

The results for $m = 4, 5$ suggest the general formula only works when n is large enough. The question becomes: is there any simple formula that works for all m and n?

4 The General Case

In general, on any GW(n, m), the centers are of degree n, and the vertices on the rim are of degree $m+2$. Let t denote the number of 0-vertices among the m centers, then the remaining $m-t$ centers are labeled 1. The numbers of 0- and 1-vertices on the rim depends on the parity of $m+n$.

If $m+n$ is even, then $v(0) = v(1) = \frac{m+n}{2}$. We summarize the numbers of 0- and 1-vertices on the rim C_n and among N_m in the following chart.

	C_n	N_m
# of 0-vertices	$\frac{m+n}{2} - t$	t
# of 1-vertices	$\frac{m+n}{2} - m + t$	$m - t$

Then

$$\sum_{f(v)=0} \deg(v) = \left(\frac{m+n}{2} - t\right)(m+2) + tn,$$

$$\sum_{f(v)=1} \deg(v) = \left(\frac{m+n}{2} - m + t\right)(m+2) + (m-t)n.$$

Therefore

$$e(0) - e(1) = \frac{(m-2t)(m+2) + (2t-m)n}{2} = \frac{(2t-m)(n-m-2)}{2}.$$

We need a careful analysis of the range of values that t can assume. We need $t \le m$ as well as $t \le v(0) = \frac{m+n}{2}$. Hence $t \le \min\left(m, \frac{m+n}{2}\right)$. Likewise, $t \ge 0$, and $m - t \le v(1) = \frac{m+n}{2}$. Thus $\max(0, \frac{m-n}{2}) \le t$.

If $m+n$ is odd, we may assume $v(0) = \frac{m+n-1}{2}$, and $v(1) = \frac{m+n+1}{2}$:

	C_n	N_m
# of 0-vertices	$\frac{m+n-1}{2} - t$	t
# of 1-vertices	$\frac{m+n+1}{2} - m + t$	$m - t$

Then

$$\sum_{f(v)=0} \deg(v) = \left(\frac{m+n-1}{2} - t\right)(m+2) + tn,$$

$$\sum_{f(v)=1} \deg(v) = \left(\frac{m+n+1}{2} - m + t\right)(m+2) + (m-t)n.$$

Therefore

$$\begin{aligned} e(0)-e(1) &= \frac{(m-2t-1)(m+2)+(2t-m)n}{2} \\ &= \frac{(2t-m)(n-m-2)-(m+2)}{2}, \end{aligned}$$

where $\max(0, \frac{m-n-1}{2}) \le t \le \min\left(m, \frac{m+n-1}{2}\right)$. Combining these results, we obtain our main theorem.

Theorem 4.1 *For all integers $n \ge 3$, and $m \ge 1$, the balance index set of $GW(n,m)$ is*

$$\begin{cases} \left\{\left|\frac{(2t-m)(n-m-2)}{2}\right| : \max\left(0, \frac{m-n}{2}\right) \le t \le \min\left(m, \frac{m+n}{2}\right)\right\} & \\ \hfill \text{if } m+n \text{ is even,} \\ \left\{\left|\frac{(2t-m)(n-m-2)-(m+2)}{2}\right| : \max\left(0, \frac{m-n-1}{2}\right) \le t \le \min\left(m, \frac{m+n-1}{2}\right)\right\} & \\ \hfill \text{if } m+n \text{ is odd.} \end{cases}$$

References

[1] C.C. Chou and S.M. Lee, On the balance index sets of the amalgamation of complete graphs and stars, manuscript.

[2] Y.S. Ho, S.M. Lee, H.K. Ng and Y.H. Wen, On balancedness of some families of trees, *J. Combine. Math. Combin. Comput.*, to appear.

[3] S.R. Kim, S.M. Lee and H.K. Ng, On balancedness of some graph constructions, *J. Combin. Math. Combin. Comput.* **66** (2008), 3–16.

[4] H. Kwong On balance index sets of rooted trees, *Ars Combin.* **91** (2009), 373–382.

[5] H. Kwong and S.M. Lee, On balance index sets of chain sum and amalgamation of generalized theta graphs, *Congr. Numer.* **187** (2007), 21–32.

[6] H. Kwong, S.M. Lee, S.P.B. Lo, H.H. Su and Y.C. Wang, On balance index sets of L-products with cycles and complete graphs, *J. Combin. Math. Combin. Comput.* **70** (2009), 85–96.

[7] H. Kwong, S.M. Lee, S.P.B. Lo and Y.C. Wang, On uniformly balanced graphs, manuscript.

[8] H. Kwong, S.M. Lee and D.G. Sarvate, On balance index sets of one-point unions of graphs, *J. Combin. Math. Combin. Comput.* **66** (2008), 113–127.

[9] A.N.T. Lee, S.M. Lee and H.K. Ng, On the balance index sets of graphs, *J. Combin. Math. Combin. Comput.* **66** (2008), 135–150.

[10] A.N.T. Lee, S.M. Lee, S.P. Bill Lo and H.K. Ng, On balance index sets of Halin graphs of stars and double stars, manuscript.

[11] S.M. Lee, A. Liu and S.K. Tan, On balanced graphs, *Congr. Numer.* **87** (1992), 59–64.

[12] S.M. Lee, H.K. Ng and S.M. Tong, On the balance index set of the chain-sum graphs of cycles, *Util. Math.* **77** (2008), 113–123.

[13] S.M. Lee and H.H. Su, On the balance index set of the permutation graphs, manuscript.

[14] S.M. Lee, Y.C. Wang and Y.H. Wen, On the balance index sets of the $(p, p+1)$-graphs, *J. Combin. Math. Combin. Comput.* **62** (2007), 193–216.

[15] M.A. Seoud and A.E.I. Abdel Maqsoud, On cordial and balanced labelings of graphs, *J. Egyptian Math. Soc.* **7** (1999) 127–135.

[16] W.C. Shiu and H. Kwong, An algebraic approach for finding balance index sets, *Australas. J. Combin.* **45** (2009), 139–155.

[17] D.H. Zhang, Y.S. Ho, S.M. Lee and Y.H. Wen, On balance index sets of trees with diameter at most four, *J. Combin. Math. Combin. Comput.*, to appear.