Vertex-Magic Edge Labeling Games on Graphs with Cycles

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Abstract

Given a graph G, let E be the number of edges in G. A *vertex-magic edge labeling* of G, defined by Wallis [12] in 2001, is a one-to-one mapping from the set of edges onto the set $\{1, 2, \ldots, E\}$ with the property that at any vertex the sum of the labels of all the edges incident to that vertex is the same constant. In 2003, Hartnell and Rall [5] introduced a two player game based on these labelings, and proved some nice results about winning strategies on graphs that contain vertices of degree one. In this paper we prove results about winning strategies for certain graphs with cycles where the minimum degree is two.

1 Introduction

Since ancient times, objects called magic squares served as a source of mathematical recreation [12]. By definition, a *magic square* of side n is an $n \times n$ array with entries taken from the set $\{1, 2, \ldots, n^2\}$ and with the property that the sum of the elements in each row, column and main diagonal is constant. In 1963, Sedlâček introduced the idea of a magic labeling of graphs [8], and soon afterwards, Stewart began researching ways to label graph edges to answer some of Sedlâček's questions [9],[10]. A *graph labeling* is defined to be a one-to-one mapping from a set of graph elements, usually vertices and/or edges, onto a set of positive integers. In particular, a labeling of the set of edges of a graph with consecutive positive integers such that the sum of the labels of all the incident edges to a vertex is constant, is called a *supermagic labeling.* Note that this definition gives a correspondence between the $n \times n$ magic square and a supermagic labeling of $K_{n,n}$ [9].

In 2002, MacDougall, Miller, Slamin and Wallis defined *vertex-magic total labeling* to be a labeling from the set of vertices and edges of a graph to the set $\{1, 2, \ldots, V+E\}$ (where V is the number of vertices in the graph and E is the number of edges in the graph) with the property that at any vertex the sum of the label of that vertex together with all the labels of the edges incident to that vertex is constant [7]. By restricting the labeling to edges only, Wallis introduced the following definition: a *vertex-magic edge labeling* is a labeling from the set of edges to the set $\{1, 2, \ldots, E\}$, with the property that at any vertex the sum of the labels of all the edges incident to that vertex is the same constant (called the *magic constant*) [12]. Many have studied which graphs admit various kinds of magic labelings, what conditions are necessary or sufficient for graphs to have magic labelings, and how to construct such labelings. For a survey of results on magic labeling (and graph labeling in general), see [3], which quotes well over one hundred articles on magic labeling alone.

In 2002, Boudreau, Hartnell, Schmeisser and Whitely introduced the notion of a two player game based on vertex-magic total labelings of graphs [1],[2]. The two players alternate turns, each labeling a single edge or vertex of G so that none of the requirements for a vertex-magic total labeling are violated (no label is repeated and when any two vertices and all of the edges incident to those vertices have been labeled the labels add to the same constant). The player able to make the last legal move on G wins the game.

Given a graph, let the induced subgraph on the vertices of degree greater than or equal to two be called the *core.* Call a vertex of degree one a *leaf* and a vertex adjacent to a vertex of degree one a *stem.* Let G be a graph in which every vertex is a leaf or a stem. Boudreau, Hartnell, Schmeisser and Whitely proved that when playing the vertex-magic total labeling game, if G also has certain restrictions on the stems, Player 1 has a winning strategy if the core has an odd number of edges and Player 2 has a winning strategy otherwise [1]. They also considered graphs where at most one vertex was not a leaf or a stem, showing that Player 1 has a winning strategy when there are certain restrictions on the leaves [1], [2].

In 2003, Hartnell and Rall applied vertex-magic edge labelings of graphs to the game [5] (the players only label edges). They proved what determines whether Player 1 or Player 2 has a winning strategy on graphs where every vertex is a leaf or a stem, both when the core is even and when the core is odd [5]. They also proved that Player 1 has a winning strategy in graphs where the vertices are either leaves, stems or vertices of degree two, the core can be partitioned into paths between stems with certain restrictions, and the number of edges in G is odd. Tesmenitsky then investigated graphs with similar properties but where the number of edges in G is even [11]. Note that the graphs in all of these results are required to have vertices of degree one. Building on the work of Hartnell, Rall and Tesmenitsky, we prove results about the vertex-magic edge labeling game on other graphs.

The following notation and terminology was introduced by Boudreau, Hartnell, Schmeisser and Whitely [1], and adapted by Hartnell and Rall [5]. We will use Hartnell and Rall's adaptation since we will be using vertex-magic edge labeling (as opposed to vertex-magic total labeling). Let $S(v)$ denote the set of edges incident with v. If, given a specific vertex v of G, we have that not every edge of $S(v)$ has received a label, then we call this a *partial labeling* of v and we call the sum of the labels of the edges of $S(v)$ the *partial weight* of v. However, if all the edges of $S(v)$ have received a label, then we call the sum of the labels of $S(v)$ the *weight* of v. The first time that all of the edges incident with a vertex in G are labeled, we call the sum of the labels the *magic constant* for G, which we will assign the letter k. Once the magic constant has been set, the weights of the remaining vertices of G must equal the magic constant k. If this is not possible at a vertex v, then at least one edge incident with v must be left unlabeled and v must remain partially labeled. Note that the partial weight of a given vertex may exceed k.

Before proceeding with our results, we will introduce some additional terminology, some due to Boudreau, Hartnell, Schmeisser and Whitely [1] and some of our own. If the magic constant k has been set and a pair x, y of edge labels adds to k, then y is called the *match* for x and vice versa. An unlabeled edge is called *eligible* if there exists a label that has not yet been used that can be used to label that edge. Otherwise, we call the edge *ineligible.* Note that the definition of an eligible edge has to do with the existence of a label that can be used on that edge. Let us define an edge to be a *free edge* if the given edge can receive ANY of the remaining labels. Note that every free edge is an eligible edge, but not every eligible edge is necessarily a free edge. If every edge in a path is free, we will call the path a *free path.* Similarly, a *free cycle* is a cycle in which all the edges are free.

In previous results, the key to the strategies generally revolved around Player 1 labeling a vertex of degree one. We chose to investigate graphs in which there are no vertices of degree one, thus necessitating new strategies. First we prove results about cycles, and then we prove results about graphs formed by taking two cycles and identifying one of their vertices. This work is based on work done in Giambrone's honors thesis [6].

2 Cycles

Our first results are for playing the vertex-magic edge labeling game on single cycles. After the initial move is made on a cycle, what remains to be labeled is a path. Hence, in order to prove who may have the winning strategy on cycles, we first devise strategies for free paths when these paths are encountered while the game is already in progress (in particular, the magic constant has already been set).

Suppose we have a free path, P, of even length $2n$. Let the edges of P be called e_1 through e_{2n}. Also suppose the remaining edge labels may be paired into matches. After the first player labels an edge, the second player to play on P may use the following strategy to ensure that what remains after his/her response are disjoint free paths of even length:

Strategy 1 *If the first player plays on e_1 or e_{2n}, then the second player responds with the match on the other. If the first player plays on e_{2k} for $0 < k < n$, then the second player responds with the match on e_{2k+1}. If the first player plays on e_{2k+1} for $0 < k < n$, then the second player responds with the match on e_{2k}.*

The consequence of Strategy 1 is exactly four edges (or, in the case when $n = 1$, the two edges that make up the entire path) are eliminated (labeled or made ineligible) from P after the first two moves on P, and this is done in such a way that what remains are no, one or two disjoint free paths of even length. In the first case, this is done by eliminating two edges from each end of P. In the other two cases, this is done by eliminating the edges e_{2k-1}, e_{2k}, e_{2k+1} and e_{2k+2}, leaving paths from e_1 to e_{2k-2} and e_{2k+3} to e_{2n}. This approach will often prove effective as seen in the proof of the following lemma.

Note that as we prove the lemmas, we will refer to a Player 1 and a Player 2. These titles will be in relationship to the situation of the lemma and not necessarily to the game as a whole. For example, in the proof of Lemma 2, Player 1 refers to the first player to play on the $2n$ edges of the free path, which may not be the player who started the game.

Lemma 2 *Assume that a graph G has been partially labeled and that the magic constant has been set. Suppose that the unplayed labels can all be paired so that each pair of labels adds to the magic constant. Also suppose the remaining playable portion of G, call it G', is a set of disjoint, induced free paths of even lengths. Then there is a strategy for the second player to play on this set of paths to make the last legal move.*

Proof. Assume the hypotheses of the lemma hold. We will proceed by induction on the number, $2n$, of edges in G'. Call the first player to play

Player 1 and the second player to play Player 2. Since every label has a match, clearly Player 2 will win if $n = 1$, that is, if there is exactly one free path of length two remaining.

Suppose $n > 1$. After Player 1 plays on an edge of one of the free paths, P, Player 2 responds on P using Strategy 1. According to this strategy, either two edges (if P has length two) or four edges have been eliminated after Player 2's response. Hence the remaining graph has fewer edges, and and also by Strategy 1, it maintains the property that all remaining edges are in disjoint, induced free paths of even length. Thus by induction Player 2 can make the last legal move.

Therefore, Player 2 has a winning strategy on free paths of even length. □

Using Lemma 2, we may now prove the following result about cycles of even length.

Theorem 3 *For $n \geq 2$, Player 2 has a winning strategy on the cycle C_{2n}, regardless of the initial move of Player 1.*

Proof. After Player 1 plays the first label, call it m, on an edge of the cycle C_{2n}, Player 2 responds by playing the label $(2n+1) - m$ on an edge adjacent to the one played on by Player 1. Hence, the magic constant has been set to be $2n + 1$, and the unlabeled edges adjacent to the two labeled edges have become ineligible. Since there are $2n$ edges in the cycle, then there are an even number of remaining labels and these labels can be paired with their matches in the following manner: $(1, 2n), (2, 2n-1), \ldots, (m-1, 2n+2-m), (m+1, 2n-m), \ldots, (n, n+1)$.

If $n = 2$, then no edges of C_{2n} remain that are eligible to be labeled. If $n > 2$, then an induced free path of nonzero even length $2n - 4$ has been created. Since Player 2 will be the second person to play on this path, then, by Lemma 2, Player 2 is able to make the last legal move on this path, and hence has a winning strategy for the original even length cycle. □

Next we turn our attention to odd cycles. We split this into two cases: cycles of length 1 modulo 4 and cycles of length 3 modulo 4. In order to prove the result for cycles of length 1 modulo 4, we first establish a strategy for free paths of length 3 modulo 4.

Lemma 4 *Assume that a graph G has been partially labeled and that the magic constant has been set. Suppose that the remaining labels can all be paired so that each pair of labels adds to the magic constant, and suppose the set of remaining eligible edges forms an induced free path, P, of length $4n + 3$ where $n \geq 0$. Then there is a strategy for the second player to play to make the last legal move on P.*

Proof. Assume the hypotheses of the lemma hold. Call the first player to play Player 1 and the second player to play Player 2. We will proceed by induction on n. Suppose that P is a path of length three. Since each remaining label has a match, no matter what edge of P Player 1 plays on, Player 2 can respond on either of the two remaining edges of P, making the final edge ineligible. Therefore, Player 2 can make the last legal move if P has length three.

Assume $n > 0$, that is, P has length greater than three. Name the edges of the path e_1 through e_{4n+3}. Since the remaining labels can be paired with their matches, the strategy of Player 2 will be to respond to Player 1's move by labeling an edge with the match.

Suppose Player 1 plays on e_{4k} of P, where $1 \leq k \leq n$. Then Player 2 may respond with the match on e_{4k+2}, thus the edges e_{4k-1}, e_{4k+1}, and e_{4k+3} become ineligible. This leaves two induced free paths, one of length $4k-2$ from e_1 to e_{4k-2}, and one of length $4(n-k)$ (possibly zero) from e_{4k+4} to e_{4n+3}. Since both of these paths have even length, by Lemma 2, Player 2 can make the last legal move on these paths.

Suppose Player 1 plays on e_{4k+1} or e_{4k+2} of P, where $0 \leq k \leq n$. Then Player 2 may respond with the match on the other, thus making the edges e_{4k} (if it exists, i.e. $k > 0$) and e_{4k+3} ineligible. This leaves two induced free paths, one of length $4k-1$ (or 3 modulo 4) from e_1 to e_{4k-1}, and one of length $4(n-k)$ from e_{4k+4} to e_{4n+3}. If $k = 0$ or $k = n$, only one of these paths will have nonzero length. Regardless, Player 2 can make the last legal move on these paths: on the first by induction and the second by Lemma 2.

Suppose Player 1 plays on e_{4k+3} of P, where $0 \leq k \leq n$. Then Player 2 may respond with the match on e_{4k+2}, thus making the edges e_{4k+1} and e_{4k+4} (if it exists, i.e. $k < n$) ineligible. This leaves two induced free paths, one of length $4k$ from e_1 to e_{4k}, and one of length $4(n-k)-1$ (or 3 modulo 4) from e_{4k+5} to e_{4n+3}. If $k = 0$ or $k = n$, only one of these paths will have nonzero length. Again, Player 2 can make the last legal move on these paths: on the first by Lemma 2 and the second by induction.

Therefore, Player 2 can always make the last legal move on a free path of length $4n+3$ for any $n \geq 0$. □

Note that if at any point in a game after the magic constant has been set the remaining graph is a set of induced free paths of even length or length 3 modulo 4, the second player to play on this set has a winning strategy. This player simply treats the paths as a set of simultaneous games, using Lemmas 2 and 4 where appropriate. This idea may be extended throughout the paper as we prove further results. The next lemma illustrates a strategy for paths to be played on in pairs.

Lemma 5 *Assume that a graph G has been partially labeled and that the magic constant has been set. Suppose the number of remaining labels that are nonmatchable is even (possibly zero). Also suppose the set of remaining eligible edges forms a pair of induced free paths of the same length. Then there is a strategy for the second player to play to make the last legal move on this pair of isomorphic paths.*

Proof. Assume the hypotheses of the lemma hold. Since the number of remaining nonmatchable labels is even, the total remaining labels is even. Call the first player to play Player 1 and the second player to play Player 2. After Player 1 labels an edge of one of the paths, Player 2 may respond by playing on the corresponding edge of the other path, essentially mimicking the move made by Player 1. If the label Player 1 used has a match, Player 2 responds with the match. If the label Player 1 used does not have a match, Player 2 responds with a nonmatchable label. Since isomorphic edges of the two paths will be eliminated each time, when Player 1 makes the last legal move on one of the paths, Player 2 responds and makes the last legal move on the second path and, therefore, makes the last legal move on the pair of paths. □

The above lemmas give us the ability to make conclusions about cycles of length 1 modulo 4.

Theorem 6 *For any positive integer n, Player 1 has a winning strategy on the cycle C_{4n+1}.*

Proof. Let the edges of the cycle C_{4n+1} be called e_1 through e_{4n+1}. Without loss of generality, Player 1 begins by placing the label $4n+1$ on edge e_1. Then Player 2 may respond (a) on an edge adjacent to e_1, (b) on the edge e_3 (or, symmetrically, on e_{4n}), or (c) on an edge e_j where $4 \leq j \leq 4n-1$.

Suppose Player 2 responds on an edge adjacent to e_1. Without loss of generality, suppose Player 2 responds by labeling the edge e_2 with m. Hence, the magic constant has now been set to $4n+1+m$. Note the edges e_{4n+1} and e_3 are now ineligible and what remains to be labeled is a free path of length $4n-3$ from e_4 to e_{4n}. If $n=1$, then this is a free path of length one and trivially Player 1 wins. So suppose $n \geq 2$. Since the game begins with an odd number of labels and two labels have been used, there remains an odd number of labels from which to choose and thus at least one of those labels has no match. Hence Player 1 may respond by playing a nonmatchable label on the edge e_{2n+2}. Since this label has no match, the edges e_{2n+1} and e_{2n+3} become ineligible. What remains to be labeled are two induced, isomorphic free paths of length $2n-3$. Specifically, the first path includes edges e_4 to e_{2n} and the second path includes edges

e_{2n+4} to e_{4n}. Now there are an even number of labels remaining. The remaining labels that can be paired with matches clearly must add to an even number. Thus, by a simple parity argument, there are also an even number of nonmatchable labels. Therefore, by Lemma 5, Player 1 can make the last legal move on the paths and, hence, on C_{4n+1}.

Suppose Player 2 responds by playing the label m on the edge e_3 or on the edge e_{4n}. Without loss of generality, assume Player 2 responds on e_3. Player 1 then may respond by playing the label $4n+1-m$ on the edge e_4, thus setting the magic constant to $4n+1$. Since e_1 is labeled $4n+1$, e_2 and e_{4n+1} are now ineligible, in addition to e_5. If $n = 1$, then no edges remain to be labeled and Player 1 has made the last legal move. Assume $n \geq 2$. Since three labels have been used, there are an even number of remaining labels and these labels can be paired with their matches in the following manner: $(1, 4n), (2, 4n-1), \dots, (m-1, 4n-m+2), (m+1, 4n-m), \dots, (2n, 2n+1)$. Now six adjacent edges have either been labeled or been made ineligible, so what remains to be labeled is an induced free path of length $4n-5$ (equivalent to 3 modulo 4) from e_6 to e_{4n}. Hence, by Lemma 4, Player 1 can make the last possible move.

Suppose Player 2 responds by playing the label m on the edge e_j where $4 \leq j \leq 4n-1$. Note that this implies $n \geq 2$. Player 1 may respond by placing the label $4n+1-m$ on an edge adjacent to e_j as prescribed by Strategy 1. This sets the magic constant to $4n+1$. Since e_1 is labeled $4n+1$, e_2 and e_{4n+1} are now ineligible. Consider the "free" path of even length $4n-2 \geq 6$ of edges from e_3 to e_{4n}. Since Player 1 is the second player to play on this path and s/he followed Strategy 1, what remains to be labeled from this path after Player 1's move is either a single free path of even length or a pair of free paths of even length. Therefore, by Lemma 2, Player 1 is able to make the last legal move.

Therefore, considering all cases, Player 1 has a winning strategy on a cycle of length $4n+1$, where $n \geq 1$. □

In order to extend our results to cylces of length 3 modulo 4, we must have information about free paths of length 1 modulo 4 that are encountered when the game is in progress and the magic constant has been set. Note that to prove our results about paths of even length and length 3 modulo 4 we employed induction. Unfortunately, this method will not directly work for paths of length 1 modulo 4. Clearly, Player 1 will win on a path of length one. Player 1 can also win on a path of length five by making his/her first move on the middle edge (there will be exactly two remaining moves, one on each side of this edge). However, we will show that Player 2 can win on all paths of length $4n+1$ where $n \geq 2$. To show this, we consider pairs of paths where at least one of the paths in the pair is of length 1 modulo 4.

Lemma 7 *Assume that a graph G has been partially labeled and that the magic constant has been set. Suppose that the remaining labels can all be paired so that each pair of labels adds to the magic constant, and suppose the set of remaining eligible edges forms two disjoint free paths of lengths $4m+1$ and $4n+1$, respectively, where $m, n \geq 0$. Then there is a strategy for the second player to play on this pair of paths to make the last legal move.*

Proof. Assume the hypotheses of the lemma hold. Call the first player to play on this pair of paths Player 1 and the second player Player 2. We will proceed by induction on $m+n$. Suppose that $m+n=0$. Then P_{4m+1} and P_{4n+1} are both paths of length one. When Player 1 plays on one of these paths, Player 2 responds by playing on the other. Since both paths are of length one, no edges remain. Therefore, Player 2 can make the last legal move if $m+n=0$.

Assume $m+n>0$. Then at least one of P_{4m+1} and P_{4n+1} has length greater than one. Let the edges of the path P_{4m+1} be called e_1 through e_{4m+1}, and the edges of the path P_{4n+1} be called f_1 through f_{4n+1}. Since the two paths both have length 1 modulo 4, we may assume without loss of generality that Player 1 makes her/his first move on P_{4m+1}. Since the remaining labels can be paired with their matches, the strategy of Player 2 will be to respond to Player 1's move on P_{4m+1} by labeling an edge on P_{4n+1} with the match. Player 2 will do this in such a way that the subscripts of the two players' edges are equivalent modulo 4 if $n \geq 1$, that is, if P_{4n+1} has length greater than 1. If P_{4n+1} has length one, then Player 2 will simply label f_1.

Suppose Player 1 plays on e_{4k+1} of P_{4m+1}, where $0 \leq k \leq m$. (Note that if $m=0$ this is the only possible move on P_{4m+1}.) Then Player 2 may respond with the match on f_{4s+1}, where $0 \leq s \leq n$. Then the edges e_{4k}, e_{4k+2}, f_{4s} and f_{4s+2} (if they exist) become ineligible. This leaves zero (if the path has length one, which can be true for at most one of P_{4m+1} and P_{4n+1}), one or two induced free paths of length 3 modulo 4 from each of the original paths. Since all of these paths have length 3 modulo 4, by Lemma 4, Player 2 can make the last legal move on each of these paths.

Suppose Player 1 plays on e_{2k} of P_{4m+1}, where $k>0$. Then clearly $m>0$. If $n>0$ Player 2 may respond with the match on f_{2s} where $s>0$, and if $n=0$ Player 2 may respond on f_1. The edges e_{2k-1}, e_{2k+1}, f_{2s-1} and f_{2s+1} (if they exist) then become ineligible. This leaves one or two induced free paths of even length from each of the original paths if $n>0$, and such paths only from P_{4m+1} if $n=0$. Hence Player 2 can make the last legal move on these paths by Lemma 2.

Suppose Player 1 plays on e_{4k+3} of P_{4m+1}, where $0 \leq k < m$. Then clearly $m>0$. If $n>0$ Player 2 may respond with the match on f_{4s+3} of

P_{4n+1} where $0 \leq s < n$, and if $n = 0$ Player 2 may respond on f_1. Thus the edges e_{4k+2}, e_{4k+4}, f_{4s+2} and f_{4s+4} (if they exist) become ineligible. This leaves two induced free paths of length 1 modulo 4 from P_{4m+1} and zero or two such paths from P_{4n+1}. Let the two paths from P_{4m+1} be called Q_1 and Q_2, and the two paths from P_{4n+1}, if they exist, be called Q'_1 and Q'_2. Clearly the sum of the lengths of Q_1 and Q_2 is less than $m + n$, as is the sum of the lengths of Q'_1 and Q'_2, if they exist. Then, considering the pairs Q_1 and Q_2, and Q'_1 and Q'_2, Player 2 can make the last legal move by induction.

Therefore, considering all cases, Player 2 is able to make the last legal move on a pair of paths each of which is of length 1 modulo 4. □

Note that if the two players were presented with a path of length one together with a path of length three, then Player 1 could win in the following manner: Player 1 would begin by labeling one of the end edges of the path of length three. Then there would be two moves remaining – one on each path. Hence, this combination of lengths must be eliminated from the following lemma.

Lemma 8 *Assume that a graph G has been partially labeled and that the magic constant has been set. Suppose that the remaining labels can all be paired so that each pair of labels adds to the magic constant, and suppose the set of remaining eligible edges forms two disjoint free paths, the first of length one and the second of finite nonzero length, such that if the second path is of length 3 modulo 4, then it must be of length $4k + 3$ where $k \geq 1$. That is, the second path may not have length three. Then there is a strategy for the second player to play on this pair of paths to make the last legal move.*

Proof. Assume the hypotheses of the lemma hold. Let Player 1 be the first player to play on the pair of paths and Player 2 be the second. Denote the first path of length one by P_1 and denote the sole edge of P_1 by e^*. Denote the second path by Q, and call Q's edges $e_1, e_2, \ldots, e_n$, in order. Note that Q may be of nonzero length 0 modulo 4, 1 modulo 4, 2 modulo 4, or 3 modulo 4 with the stipulation that if Q has length 3 modulo 4, then it must have length 7 or greater.

If Q is of length 1 modulo 4, then by Lemma 7, Player 2 can win on this pair of paths. Thus we may assume that the second path is not of length 1 modulo 4. The first move of Player 1 may either be on e^* or an edge of Q.

Suppose the first move of Player 1 is on e^*. Then Player 2 may respond with the match on e_1 (making e_2 ineligible) if Q has length 0 modulo 4 or 2 modulo 4, and with the match on e_2 (making e_1 and e_3 ineligible) if Q has length 3 modulo 4. In all cases, what remains of G is a free path of even length. Hence by Lemma 2, Player 2 has a winning strategy. Thus we may assume Player 1 makes his/her first move on an edge of Q. Our strategy for Player 2 varies according to the length of Q modulo 4.

Suppose that Q is of nonzero length 0 modulo 4. If Player 1 plays on one of e_{4k+1} and e_{4k+2}, where $k \geq 0$, then Player 2 may respond with the match on the other, making e_{4k} (if it exists) and e_{4k+3} ineligible. Then what remains is a path from e_1 to e_{4k-1} of length 3 modulo 4 (if $k > 0$ and of length 0 if $k = 0$), a path from e_{4k+4} to e_n of length 1 modulo 4, and P_1. By Lemma 4, Player 2 has a winning strategy for this first path, and by Lemma 7 Player 2 has a winning strategy when considering the other two paths as a pair.

If Player 1 plays on one of e_{4k+3} and e_{4k+4}, where $k \geq 0$, then Player 2 may respond with the match on the other, making e_{4k+2} and e_{4k+5} (if it exists) ineligible. Then what remains is a path from e_{4k+6} to e_n of length 3 modulo 4 (if $k > 0$ and of length 0 if $k = 0$), a path from e_1 to e_{4k+1} of length 1 modulo 4, and P_1. By Lemma 4, Player 2 has a strategy for this first path, and by Lemma 7 Player 2 has a winning strategy when considering the other two paths as a pair. Hence, Player 2 has a winning strategy when Q is of length 0 modulo 4.

Suppose that Q is of length 2 modulo 4. If Player 1 plays on one of e_{4k+1} and e_{4k+2}, where $k \geq 0$, then Player 2 may respond with the match on P_1, making one (if the move is on e_1 or e_n) or two edges ineligible on Q. Then what remains are two free paths, one of length 3 modulo 4, and one of even length, either possibly of length zero. By Lemma 4, Player 2 has a winning strategy for the path of length 3 modulo 4, and by Lemma 2 Player 2 has a winning strategy for the even length path.

If Player 1 plays on e_{4k+3} where $k \geq 0$, then Player 2 may respond with the match on e_{4k+5}, making e_{4k+2}, e_{4k+4} and e_{4k+6} ineligible. Then what remains is a path from e_{4k+7} to e_n of even length (possibly zero), a path from e_1 to e_{4k+1} of length 1 modulo 4, and P_1. By Lemma 2, Player 2 has a winning strategy for this first path if it exists, and by Lemma 7 Player 2 has a winning strategy when considering the other two paths as a pair.

If Player 1 plays on e_{4k} where $k \geq 1$, then Player 2 may respond with the match on e_{4k-2}, making e_{4k-3}, e_{4k-1} and e_{4k+1} ineligible. Then what remains is a path from e_1 to e_{4k-4} of even length (possibly zero), a path from e_{4k+2} to e_n of length 1 modulo 4, and P_1. By Lemma 2, Player 2 has a strategy for this first path if it exists, and by Lemma 7 Player 2 has a winning strategy when considering the other two paths as a pair. Hence, Player 2 has a winning strategy when Q is of length 2 modulo 4.

Suppose that Q is of length 3 modulo 4, i.e. $n = 4q + 3$. By the symmetry of Q, we need only consider edges e_1 through e_{2q+2}. (We could have made similar assumptions in the past as well, but it was unnecessary.) If Player 1 plays on one of e_{4k+1} and e_{4k+2}, where $k \geq 0$ and $4k+2 \leq 2q+2$, then Player 2 may respond with the match on the other, making e_{4k} (if it exists) and e_{4k+3} ineligible. Then what remains is a path from e_1 to e_{4k-1} of length 3 modulo 4 (possibly zero), a path from e_{4k+4} to e_n of even length, and P_1. By Lemma 4, Player 2 has a winning strategy for this first path if it exists, and by the 0 modulo 4 case in this proof Player 2 has a winning strategy when considering the other two paths as a pair.

If Player 1 plays on e_{4k+3} where $k \geq 0$, then Player 2 may respond with the match on e_{4k+4}, making e_{4k+2} and e_{4k+5} ineligible. Then what remains is a path from e_{4k+6} to e_n of even length, a path from e_1 to e_{4k+1} of length 1 modulo 4, and P_1. By Lemma 2, Player 2 has a winning strategy for this first path, and by Lemma 7 Player 2 has a winning strategy when considering the other two paths as a pair.

If Player 1 plays on e_{4k} where $k \geq 1$, then Player 2 may respond with the match on e_{4k+1}, making e_{4k-1} and e_{4k+2} ineligible. Then what remains is a path from e_1 to e_{4k-2} of even length, a path from e_{4k+3} to e_n of length 1 modulo 4, and P_1. By Lemma 2, Player 2 has a winning strategy for this first path, and by Lemma 7 Player 2 has a winning strategy when considering the other two paths as a pair. (Note that this case also offers an alternate strategy from the one above for Player 2 to respond when Player 1 labels e_{4k+1}.) Hence, Player 2 has a winning strategy when Q is of length 3 modulo 4.

Therefore, considering all cases, Player 2 is able to make the last legal move on a pair of free paths, one of which is of length one and the other of which is of nonzero length n such that $n \neq 3$. □

Lemma 9 *Assume that a graph G has been partially labeled and that the magic constant has been set. Suppose that the remaining labels can all be paired so that each pair of labels adds to the magic constant, and suppose the set of remaining eligible edges forms two disjoint free paths, the first of length $4n+1$ for some $n \geq 1$ and the second of any finite nonzero length. Then there is a strategy for the second player to play on this pair of paths to make the last legal move.*

Proof. Assume the hypotheses of the lemma hold. Denote the edges of P_{4n+1} by $e_1, e_2, \ldots, e_{4n+1}$, in order. Let the other path be of length m and denote the edges of P_m by $f_1, f_2, \ldots, f_m$, in order. Call the first player to play on this pair of paths Player 1 and the second player Player 2. If m is equivalent to 1 modulo 4, then by Lemma 7, Player 2 can make the last legal move. Thus we may assume that m is not equivalent to 1 modulo 4.

If Player 1 makes her/his first move on P_{4n+1}, then Player 2 will respond with the match on P_m. If m is even, Player 2 will respond on f_1, making f_2 ineligible. If m is equivalent to 3 modulo 4, then Player 2 will respond on f_2, making both f_1 and f_3 ineligible. In either case, the eligible edges of what remains of P_m form an even free path (possibly of length zero) and Player 2 can make the last legal move on this path by Lemma 2. Now we consider what remains of P_{4n+1}.

Suppose Player 1 played on e_{4k+1} where $0 \leq k \leq n$. Then e_{4k} and e_{4k+2}, if they exist, become ineligible. What will remain to be labeled from P_{4n+1} are one (if $k = 0$ or $k = n$) or two free paths, both of which are of length 3 modulo 4. Thus by Lemma 4, Player 2 can make the last legal move on P_{4n+1}.

Suppose Player 1 played on e_{2k} for some $1 \leq k \leq 2n$. Then e_{2k-1} and e_{2k+1} become ineligible. What will remain to be labeled from P_{4n+1} are one (if $k = 1$ or $k = 2n$) or two free paths of even length. Thus, Player 2 can make the last legal move on what remains from P_{4n+1} by Lemma 2.

Suppose Player 1 played on e_{4k+3} for $0 \leq k < n$. Then, e_{4k+2} and e_{4k+4} become ineligible. What will remain to be labeled from P_{4n+1} are two free paths, both of length 1 modulo 4. Hence, by Lemma 7, Player 2 can make the last legal move on P_{4n+1}.

Hence, if Player 1 moves first on P_{4n+1}, Player 2 will be able to make the last move.

Suppose Player 1 moves first on P_m. Recall if m is equivalent to 1 modulo 4, we are done. Thus we must consider if m is even and if m is equivalent to 3 modulo 4. Suppose m is even. We will proceed by induction on m. Suppose that P_m is a path of length two. After Player 1 makes the first move on P_m, Player 2 may responds with the match on e_1 of P_{4n+1}. Then e_2 becomes ineligible, no edges remain from P_m, and what remains from P_{4n+1} is either no edges (if $n = 0$) or a free path of length 3 modulo 4 (if $n \geq 1$). Then, by Lemma 4, Player 2 can make the last legal move on P_{4n+1}.

Assume $m > 2$, that is, P_m has even length greater than two. If $m = 4$ and Player 1 plays on one of f_1 and f_2 (or symmetrically f_3 and f_4), then Player 2 responds with the match on the other, thus making f_3 (or f_2) ineligible in the process. Hence, what remains from P_m is a free path of length 1 consisting of the edge f_4 (or f_1). Then, pairing this with the free path P_{4n+1}, Player 2 can make the last legal move by Lemma 7.

Now suppose $m \geq 6$. After Player 1 makes the first move on P_m, Player 2 may respond with the match according to Strategy 1. Since $m \geq 6$, after the response of Player 2, what will remain to be labeled from P_m will be one or two free paths of even length. Thus there will be a path of even length to pair with P_{4n+1}, and by induction, Player 2 can make the last legal move on this pair. If there is a second path of even length, Player 2 can make the last legal move on that path by Lemma 2. Hence, if m is even, Player 2 can make the last legal move.

Suppose that m is equivalent to 3 modulo 4, that is, $m = 4k + 3$ for some $k \geq 0$. By symmetry, we need only consider the first $2k + 2$ edges of P_m. Suppose Player 1 makes the first move on f_{4q+1} or f_{4q+2}, where $0 \leq 4q \leq 2k$. Then Player 2 may respond with the match on the other, making the edges f_{4q} (if it exists) and f_{4q+3} ineligible. This leaves one (if $q = 0$) or two induced free paths, the first from f_1 to f_{4q-1} of length 3 modulo 4 (if it is of nonzero length), and the second from f_{4q+4} to f_{4k+3} of even length. If the first of these exists, Player 2 can make the last legal move on it by Lemma 4. The second path can be paired with P_{4n+1} and by the previous case, Player 2 can make the last legal move on this pair. Thus, if Player 1 makes the first move on f_{4q+1} or f_{4q+2}, Player 2 can win.

Suppose Player 1 makes the first move on f_{4q+3} or f_{4q+4}, where $0 \leq 4q + 2 \leq 2k$. Then Player 2 may respond with the match on the other, making the edges f_{4q+2} and f_{4q+5} ineligible. This leaves two induced free paths, the first from f_1 to f_{4q+1} of length 1 modulo 4, and the second from f_{4q+6} to f_{4k+3} of even length. By Lemma 2, Player 2 can make the last legal move on the second of these. The first path can be paired with P_{4n+1} and by Lemma 7, Player 2 can make the last legal move on this pair. Thus, if Player 1 makes the first move on f_{4q+3} or f_{4q+4}, Player 2 can win. Hence, Player 2 can win if m is equivalent to 3 modulo 4.

Therefore, considering all cases, Player 2 is able to make the last legal move on a pair of paths one of which is of length $4n + 1$ for some $n \geq 1$ and the second is of any finite nonzero length. □

Combining Lemma 8 and Lemma 9, we have the following.

Corollary 10 *Assume that a graph G has been partially labeled and that the magic constant has been set. Suppose that the remaining labels can all be paired so that each pair of labels adds to the magic constant, and suppose the set of remaining eligible edges forms two disjoint free paths, the first of length 1 modulo 4 and the second of any finite nonzero length, such that the pair of paths is not a path of length one together with a path of length three. Then there is a strategy for the second player to play on this pair of paths to make the last legal move.*

By using these results about pairs of paths, we may now prove our result for paths of length 1 modulo 4.

Lemma 11 *Assume that a graph G has been partially labeled and that the magic constant has been set. Suppose that the remaining labels can all be paired so that each pair of labels adds to the magic constant, and suppose the set of remaining eligible edges forms an induced free path P of length $4n + 1$ for $n \geq 0$. Then there is a strategy for the first player to play to* *make the last legal move on P if $0 \leq n \leq 1$, and a strategy for the second player if $n \geq 2$.*

Proof. Assume the hypotheses of the lemma hold. Call the first player to play on this path Player 1 and the second player Player 2. As discussed in the paragraph before Lemma 7, Player 1 has a winning strategy if P is of length one or five. Thus assume that $n \geq 2$. Let the edges of P be named e_1 through e_{4n+1}. By symmetry, we need only consider the first $2n+1$ edges of P.

Suppose Player 1 plays first on one of e_{4k+1} and e_{4k+2} where $0 \leq 4k \leq 2n$. Then Player 2 responds with the match on the other. Thus e_{4k} (if it exists) and e_{4k+3} become ineligible, and what remains to be labeled is one (if $k = 0$) or two free paths, the first of length 3 modulo 4 and the second of even length. Then Player 2 can make the last legal move on the first by Lemma 4 and the second by Lemma 2. Hence Player 2 can win.

Suppose Player 1 plays first on one of e_{4k+3} and e_{4k+4} where $0 \leq 4k < 2n$. Then Player 2 responds with the match on the other. Thus e_{4k+2} and e_{4k+5} become ineligible, and what remains to be labeled are two free paths, the first of length 1 modulo 4 and the second of even length. Then Player 2 can make the last legal move on this pair of paths by Corollary 10.

Therefore, given a path of length $4n+1$, Player 1 has a winning strategy if $0 \leq n \leq 1$ and Player 2 has a winning strategy if $n \geq 2$. □

Note that this lemma will not be as easy to universally apply as the previous lemmas since paths of length one and five are not included. However, combining our results in Lemmas 2 and 4, and Lemma 11, we do have the following corollary.

Corollary 12 *Assume that a graph G has been partially labeled and that the magic constant has been set. Suppose that the remaining labels can all be paired so that each pair of labels adds to the magic constant, and suppose the set of remaining eligible edges forms a free path of length $n > 1$ such that $n \neq 5$. Then there is a strategy for the second player to play on this path to make the last legal move.*

Lemma 11 also gives us the ability to prove our result about cycles of length 3 modulo 4. The proof of this theorem mimics the proof of Theorem 6.

Theorem 13 *For $n \geq 3$, Player 1 has a winning strategy on the cycle C_{4n+3}.*

Proof. Assume the hypotheses of the theorem hold. Name the edges of the cycle e_1 through e_{4n+3}. Without loss of generality, Player 1 begins by placing the label $4n+3$ on edge e_1. Then Player 2 may respond (a) on an edge adjacent to e_1, (b) on the edge e_3 (or, symmetrically, on e_{4n+2}), or (c) on an edge e_j where $4 \leq j \leq 4n+1$.

Suppose Player 2 responds on an edge adjacent to e_1. Without loss of generality, suppose Player 2 responds by labeling the edge e_2 with m. Hence, the magic constant has now been set to $4n+3+m$. Note the edges e_{4n+3} and e_3 are now ineligible and what remains to be labeled is a free path of length $4n-1$ from e_4 to e_{4n+2}. Recall that $n \geq 3$, so this path is of length at least eleven. Since the game begins with an odd number of labels and two labels have been used, there remains an odd number of labels from which to choose and thus at least one of those labels has no match. Hence Player 1 may respond by playing a nonmatchable label on the edge e_{2n+3}. Since this label has no match, the edges e_{2n+2} and e_{2n+4} become ineligible. What remains to be labeled are two induced, isomorphic free paths of length $2n-2$. Specifically, the first path includes edges e_4 to e_{2n+1} and the second path includes edges e_{2n+5} to e_{4n+2}. Now there are an even number of labels remaining. The remaining labels that can be paired with matches clearly must add to an even number. Thus, by a simple parity argument, there are also an even number of nonmatchable labels. Therefore, by Lemma 5, Player 1 can make the last legal move on the paths and, hence, on C_{4n+3}.

Suppose Player 2 responds by playing the label m on the edge e_3 or on the edge e_{4n}. Without loss of generality, assume Player 2 responds on e_3. Then Player 1 may respond by playing the label $4n+3-m$ on the edge e_4, thus setting the magic constant to $4n+3$. Since e_1 is labeled $4n+3$, e_2 and e_{4n+3} are now ineligible, in addition to e_5. Since three labels have been used, there are an even number of remaining labels and these labels can be paired with their matches in the following manner: $(1, 4n+2), (2, 4n+1), \ldots, (m-1, 4n-m+4), (m+1, 4n-m+2), \ldots,$ $(2n+1, 2n+2)$. Now six adjacent edges have either been labeled or been made ineligible, so what remains to be labeled is an induced free path of length $4n-3$ (equivalent to 1 modulo 4) from e_6 to e_{4n+2}. Hence, since $n \geq 3$, by Lemma 11, Player 1 can make the last possible move.

Suppose Player 2 responds by playing the label m on the edge e_j where $4 \leq j \leq 4n+1$. Player 1 may respond by placing the label $4n+3-m$ on an edge adjacent to e_j as prescribed by Strategy 1. This sets the magic constant to $4n+3$. Since e_1 is labeled $4n+3$, e_2 and e_{4n+3} are now ineligible. Consider the "free" path of even length $4n \geq 12$ of edges from e_3 to e_{4n+2}. Since Player 1 is the second player to play on this path and s/he followed Strategy 1, what remains to be labeled from this path after Player 1's move is either a single free path of even length or a pair of free paths of even length. Therefore, by Lemma 2, Player 1 is able to make the last legal move.

Therefore, considering all cases, Player 1 has a winning strategy on a cycle of length $4n + 3$, where $n \geq 3$. □

Clearly, Player 2 can win on any cycle of length three. Note that Player 2 can win on a cycle of length seven, by responding to Player 1's initial move on an edge distance two away from Player 1's move. Also note that Theorem 13 can be extended to include cycles of length eleven. If we call the edge Player 1 initially labels e_1 and the remaining edges e_2 through e_{11} in clockwise order, then Player 1 can follow the strategy used in the proof of Theorem 13 when Player 2 responds on edges e_2 and e_4 through e_{11}. If Player 2 responds on e_3, then Player 1's second move should be to label e_6 (or equivalently e_9). One may check the seven resulting cases to see that Player 1 will then have a winning strategy.

Hence we have shown that Player 2 has a winning strategy on cycles of even length and length three and seven, while Player 1 has a winning strategey on cycles of odd length other than three and seven.

3 Vertex-Joined Cycle Graphs

We now generalize the idea of cycle graphs to consider a slightly more complicated family of graphs: graphs composed of two cycles with exactly one vertex in common. Given two cycles C_n and C_m of lengths n and m, respectively, identify a vertex of C_n with a vertex of C_m and call the common vertex v. (Note that v is of degree four in the new graph, while all other vertices are of degree two.) We call this new graph a *vertex-joined cycle graph* denoted by $C_n \odot C_m$. The four edges incident with v we call *junction edges.* Note that if any of the junction edges becomes ineligible, then the remaining junction edges have no restrictions at v, and so unless they have restrictions via their other endpoints, they are free edges. If such a situation occurs and no edges of C_n (or alternatively C_m) have been labeled, then, in terms of the labeling, C_n (or C_m) is equivalent to a free path of length n (or m), i.e. the cycle can be viewed as a path. In fact, the key to our strategies will be to make one of the junction edges ineligible as early in the game as possible so that we may apply the lemmas about paths from Section 2.

Our results give strategies for Player 2 to win on vertex-joined graphs $C_n \odot C_m$, where m and n are the same parity. Note if m and n are the same parity, then there are an even number of labels, which makes it possible to pair the labels.

Theorem 14 *Player 2 has a winning strategy on the graph $C_{4j+2} \odot C_{4k+2}$, for $j, k \geq 2$.*

Proof. Consider a graph $C_{4j+2} \odot C_{4k+2}$, where j and k are positive integers. Label the edges of C_{4j+2} with e_1 through e_{4j+2} consecutively, beginning and ending with a junction edge. Similarly, label the edges of C_{4k+2} with f_1 through f_{4k+2}. Without loss of generality, suppose that Player 1 makes the first move of the game on C_{4j+2} using the label l. Player 2 may respond on an adjacent edge with the label $(4j+2)+(4k+2)+1-l$, thus setting the magic constant to $4j+4k+5$. Note that the remaining labels can be paired so that each pair of labels adds to this magic constant (i.e. each label has a match). Now on which adjacent edge Player 2 makes the second move of the game is dependent on Player 1's first move. Let us consider the cases.

Suppose Player 1 makes her/his first move on e_1 (note that this is equivalent to the first move being on e_{4j+2}). Then Player 2 may respond on e_2 and e_3 becomes ineligible. What remains to be labeled from C_{4j+2} is a path P of length 3 modulo 4. Player 1 may make the second move on P or on C_{4k+2}.

If Player 1 makes his/her second move on P, then Player 2 may respond by labeling f_2 with the match, thus making f_1 and f_3 ineligible, and what remains to be labeled from C_{4k+2} is a free path, call it Q, from f_4 to f_{4k+2} of length 3 modulo 4. Note that since $k \geq 2$, this path is of length at least seven. Consider what is left of P. Label the edges of P with g_1 through g_{4w+3}. If Player 1's second move was on g_{4s+1} or g_{4t+3} for some $s, t \geq 0$, then after Player 2's second move, g_{4s} (if it exists) and g_{4s+2} or g_{4t+2} and g_{4t+4} (if it exists), respectively, become ineligible. Then what remains of P is a path of length 1 modulo 4 and possibly a path of length 3 modulo 4. Then considering the path of length 1 modulo 4 together with Q, by Corollary 10 Player 2 can make the last legal move on this pair. If the path of length 3 modulo 4 also exists, Player 2 can make the last legal move on that path by Lemma 4. So Player 2 wins. If Player 1's second move is on g_{4s} where $s \geq 1$, then g_{4s-1} and g_{4s+1} become ineligible and what remains of P are two paths of length 2 modulo 4. If Player 1's second move is on g_{4s+2} where $s \geq 0$, then g_{4s+1} and g_{4s+3} become ineligible and what remains of P are one or two paths of length 0 modulo 4. In both of these remaining cases, by using Lemma 4 on Q and Lemma 2 on whatever remains of P, Player 2 can make the last legal move to win the game.

If Player 1 makes her/his second move on C_{4k+2}, then, regardless of what edges of C_{4k+2} Player 1 labels, Player 2 will respond by labeling e_{4j+1} on C_{4j+2} with the match, thus making e_{4j+2}, a junction edge, and e_{4j} ineligible. Then what remains from C_{4j+2} is a free path P' from e_4 to e_{4j-1} of length 0 modulo 4. By Lemma 2, Player 2 can make the last legal move on P', if it exists. Note that Player 2's move also makes all eligible edges of C_{4k+2} free edges. Consider what is left of C_{4k+2}. If Player 1's second move is on f_{4s+3} or f_{4s+4} for some $s \geq 0$, then after Player

2's second move, f_{4s+2} and f_{4s+4} or f_{4s+3} and f_{4s+5}, respectively, become ineligible. Then what remains of C_{4k+2} is a path of length 1 modulo 4 and a path of length 2 modulo 4. Then considering this pair of paths, by Corollary 10 Player 2 can make the last legal move on this pair. So Player 2 wins. If Player 1's second move is on f_{4s+1} or f_{4s+2} for some $s \geq 0$, then what remains is one path of length either 0 modulo 4 or 3 modulo 4, or two paths, one of each of those lengths. In any case, by using Corollary 12 on whatever remains of C_{4k+2}, Player 2 can make the last legal move to win.

Suppose Player 1 makes the first move on either e_2 or e_3 (note that this is equivalent to the first move being on e_{4j+1} or e_{4j}). Then Player 2 may respond on the other, and e_1 (a junction edge) and e_4 become ineligible. What remains to be labeled from C_{4j+2} is a free path of length 2 modulo 4. Note that since a junction edge became ineligible, C_{4k+2} can now be considered a free path of length $4k + 2$. Thus there are two paths of even length left to be labeled. Hence by Lemma 2, Player 2 may make the last legal move and win the game.

Suppose Player 1 makes the first move on an edge e_i where $4 \leq i < 4j$. If Player 1 moves on one of e_{4q} or e_{4q+1}, then Player 2 may respond on the other, and e_{4q-1} and e_{4q+2} become ineligible. If Player 1 moves on one of e_{4q+2} or e_{4q+3}, then Player 2 may respond on the other, and e_{4q+1} and e_{4q+4} become ineligible. In either case, what remains to be labeled from C_{4j+2} are two paths, one of length 0 modulo 4 (possibly of length zero), call it P_1, and one of length 2 modulo 4, call it P_2. Let us rename the edges of P_1 with g_1 through g_{4n}, and the edges of P_2 with h_1 through h_{4m+2}, where g_1 and h_1 are junction edges. Player 1 may make his/her second move on P_1 (if it has nonzero length), P_2 or C_{4k+2}. (Note that these are not yet free paths since the included junction edges are not free edges, and so we may not apply our lemmas at this point.)

If Player 1 makes his/her second move on either P_1 or P_2, then Player 2 may respond with the match on f_2. Then f_1, a junction edge, and f_3 become ineligible. What remains to be labeled from C_{4k+2} is a free path of length 3 modulo 4, and by Lemma 4, Player 2 can make the last legal move on C_{4k+2}. Note that whatever remains of P_1 and P_2 are also free paths. Let us consider whether Player 1 moved on P_1 or P_2 to determine what the lengths of these free paths are.

If Player 1 makes her/his second move on P_1, then clearly it has nonzero length. If Player 1 labeled g_{4s+3} or g_{4s+2} where $s \geq 0$, then what remains of P_1 is a path of length 0 modulo 4 (possibly zero) and a path of length 1 modulo 4. Considering the path of length 1 modulo 4 together with P_2, Player 2 may make the last legal move on this pair of paths by Corollary 10. If the path of length 0 modulo 4 has nonzero length, Player 2 may make the last legal move on this path by Lemma 2. If Player 1 labeled g_{4s+1} or g_{4s+4} where $s \geq 0$, then what remains of P_1 is a path of length 2 modulo

4 and possibly a path of length 3 modulo 4. Then Player 2 can make the last legal move on C_{4j+2} by applying Corollary 12 to both of these paths and P_2.

If Player 1 makes her/his second move on P_2, then if P_1 has nonzero length, Player 2 can make the last legal move on P_1 by Lemma 2. If Player 1 labeled h_{4s+3} or h_{4s+4} where $s \geq 0$, then what remains of P_2 are two paths of nonzero length – one of length 2 modulo 4 and one of length 1 modulo 4. Then Player 2 may make the last legal move on this pair of paths by Corollary 10. If Player 1 labeled h_{4s+1} or g_{4s+2} where $s \geq 0$, then what remains of P_2 is a path of length 0 modulo 4, or a path of length 3 modulo 4, or one of each. Then Player 2 can make the last legal move on these paths by applying Corollary 12.

If Player 1 makes his/her second move on C_{4k+2}, then Player 2 may respond on h_2 of P_2, making h_1, a junction edge, and h_3 (if it exists – i.e. if P_2 has length greater than two) ineligible. Then either nothing remains of P_2 or a path of length 3 modulo 4 remains. Player 2 may then make the last legal move on P_1 and what remains of P_2 (if anything) by Corollary 12. We need only consider what is left of C_{4k+2}. Note that Player 2's move made any unlabeled junction edges free edges. If Player 1's second move was on f_{4s+3} or f_{4s+4} for some $s \geq 0$, then after Player 2's second move, f_{4s+2} and f_{4s+4} or f_{4s-1} and g_{4s+1}, respectively, become ineligible. Then what remains of C_{4k+2} is a path of length 1 modulo 4 and a path of length 2 modulo 4. Then considering this pair of paths, by Corollary 10 Player 2 can make the last legal move on this pair. So Player 2 wins. If Player 1's second move is on any edge of C_{4k+2} other than f_{4s+3} or f_{4s+4} for some $s \geq 0$, then what remains is one path of length 0 modulo 4, or one of length 3 modulo 4, or two paths, one of each of those lengths. In any case, by using Corollary 12 on whatever remains of C_{4k+2}, Player 2 can make the last legal move to win the game.

Therefore, Player 2 has a winning strategy on the graph $C_{4j+2} \odot C_{4k+2}$, where $j, k \geq 2$. □

Note that a case by case analysis shows that Player 2 also wins on $C_6 \odot C_6$. The strategy for Theorem 14 does not work when one of the cycles in $C_{4j+2} \odot C_{4k+2}$ is of length six, but we conjecture that Player 2 still has a winning strategy on such vertex-joined cycle graphs.

Theorem 15 *Player 2 has a winning strategy on the graph $C_{4j} \odot C_{4k}$, for $j, k \geq 2$.*

The proof of Theorem 15 is similar to that of Theorem 14. Player 2 still strives to make a junction edge ineligible in his/her first or second move. The main difference in this proof is that removing three consecutive edges from one of the cycles leaves a path of length 1 modulo 4, which could be length 5. This means that Player 2 must be careful not to try to pair a path of length one with a path of length five, and therefore additional subcases are needed.

Note that a case by case analysis shows that Player 1 has a winning strategy on the vertex-joined cycle graph $C_4 \odot C_4$. Thus although our strategy for Theorem 15 fails for vertex-joined cycle graphs of the form $C_4 \odot C_{4k}$, we conjecture there is a winning strategy for Player 1 to win on these graphs.

Using strategies similar to the proof of Theorem 14 above, we can prove the following theorems.

Theorem 16 *Player 2 has a winning strategy on the graph $C_{4j} \odot C_{4k+2}$, where j and k are positive integers.*

Theorem 17 *Player 2 has a winning strategy on the graph $C_{4j+1} \odot C_{4k+1}$, where j and k are positive integers.*

Theorem 18 *Player 2 has a winning strategy on the graph $C_{4j+3} \odot C_{4k+3}$, where $j, k \geq 0$.*

Theorem 19 *Player 2 has a winning strategy on the graph $C_{4j+1} \odot C_{4k+3}$, where $j \geq 1$ and $k \geq 0$.*

To summarize the results of the previous six theorems on vertex-joined cycle graphs, we have the following corollary.

Corollary 20 *Player 2 has a winning strategy on the graph $C_m \odot C_n$, where m and n have the same parity, and if both m and n are even, then $m, n \geq 8$.*

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