Representation numbers and Prague dimensions for complete graphs minus a disjoint union of paths

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September 1, 2010

Abstract

A graph is representable modulo n if its vertices can be assigned distinct labels from $\{0, 1, 2, \ldots, n-1\}$ such that the difference of the labels of two vertices is relatively prime to n if and only if the vertices are adjacent. The representation number $rep(G)$ is the smallest n such that G has a representation modulo n. In this paper we determine the representation number and the Prague dimension (also known as the product dimension) of a complete graph minus a disjoint union of paths.

1 Introduction

Let $G = (V, E)$ be a graph with r vertices $v_1, v_2, \ldots, v_r$. The graph G is said to have a *representation* modulo a positive integer n if there exist distinct positive integers $a_1, a_2, \ldots, a_r$ such that $0 \leq a_i < n$, and $\gcd(a_i - a_j, n) = 1$ if and only if v_i and v_j are adjacent. We say that $\{a_1, a_2, \ldots, a_r\}$ is a *representation* of G modulo n. Erdős and Evans [6] showed that every finite graph can be represented modulo some positive integer. This result was used to give a simpler proof of a result of Lindner, Mendelsohn, Mendelsohn, and Wolk [12] that any finite graph can be realized as an orthogonal Latin square graph. Narayan [14] produced a shorter proof in 2003. The representation number of a graph G, $rep(G)$, is the smallest n such that G has a representation modulo n.

The determination of $rep(G)$ for an arbitrary graph G is a very difficult problem indeed. It seems to be as difficult, if not more so, than determining $\dim_P(G)$ which has been shown to be NP-Complete [11]. Evans, Isaak, and Narayan [9] showed that the determination of representation numbers for disjoint unions of many complete graphs is dependent upon the existence of sets of mutually orthogonal Latin squares. Representation numbers for several families of graphs including complete graphs, independent sets, matchings, and graphs of the form $K_m - P_l$, $K_m - C_l$, $K_m - K_{1,l}$ (each along with a set of isolated vertices) were determined in [8] and [9]. Recently Evans [7] used linked matrices and distance covering matrices to obtain new results involving representation numbers for the disjoint union of complete graphs. Narayan and Urick [15] investigated representation numbers for split graphs, their complements, stars, and hypercubes. Recently Akhtar, Evans and Pritikin [3] produced new results involving representation numbers of stars.

Evans, Isaak, and Narayan determined the representation number of a complete graph minus a path [9]. Agarwal, Lopez and Narayan determined the representation number of a complete graph minus a disjoint union of two paths [2]. Here we determine the representation number and the Prague dimension of a complete graph minus a disjoint union of arbitrarily many paths. Note that in this family of graphs the complement $\overline{G}$ is a disjoint union of paths and possibly a set of isolated vertices.

2 Prague Dimension and Representations

A property which is closely related to the representation number of a graph G is the *Prague dimension.* The Prague dimension (also known as the product dimension) was introduced by Nešetřil and Pultr [16] and has been extensively studied [13], [4], and [5]. We say a graph G has a *product representation of length d* if each vertex v of G can be assigned a ordered d–tuple so that the vertices v and w are adjacent if and only if their vectors differ in every coordinate. The *Prague dimension* of the graph G, $\dim_P(G)$, is the minimum length d of such a representation.

As developed in [8] and [9], there is a close correspondence between Prague dimension and modular representation. Suppose G has a representation modulo n. Let $n = p_1^{m_1} p_2^{m_2} \cdots p_d^{m_d}$, where $p_1, p_2, \ldots, p_d$ are distinct primes. We obtain a product representation of G (of length d) as follows: Suppose the vertex v has label a, then the vector for v is $(v_1, v_2, \ldots, v_d)$, where $v_i \equiv a \pmod{p_i^{m_i}}$ and $0 \le v_i < p_i^{m_i}$ for $1 \le i \le d$. If vertex v with label a has vector representation $(v_1, v_2, \ldots, v_d)$ and vertex w with label b

has vector representation $(w_1, w_2, \ldots, w_d)$, then $\gcd(a-b, n) = 1$ implies that v and w are adjacent if and only if $v_i \neq w_i$ for all $1 \leq i \leq d$, making this assignment a product representation.

Now given a product representation, a modular representation can be obtained by choosing distinct primes for the coordinates, provided that the prime for each coordinate is larger than the value used in that coordinate. The numbers assigned to the vertices can then be computed using the Chinese Remainder Theorem. The resulting modular representation generated from the product representation is called the *coordinate representation.*

In [18] the question of how many prime factors, counting multiplicity, n must have for a given graph G to be representable modulo n was partially answered in terms of a type of edge labeling of the complement of G. A survey of the tools used to work on graph representations, as well as several results, can be found in [10].

3 Some known results

In this section, we restate some previously known results from [8] involving the representations modulo an integer and the representation numbers of graphs.

Theorem 1. *A graph has a representation modulo a prime if and only if it is a complete graph.*

Theorem 2. *A graph has a representation modulo a power of a prime if and only if it is a complete multipartite graph.*

The disjoint union of graphs G and H will be denoted $G + H$. That is, $V(G+H) = V(G) \cup V(H)$ and $E(G+H) = E(G) \cup E(H)$.

Theorem 3. *A graph has a representation modulo a product of some pair of distinct primes if and only if it does not contain an induced subgraph isomorphic to $K_2 + 2K_1$, $K_3 + K_1$ or the complement of a chordless cycle of length at least five.*

The following results deal with the size of the prime divisors of the representation numbers.

Theorem 4. *If G has a representation modulo n, and p is the smallest prime divisor of n then $p \geq \chi(G)$.*

We have the following corollary where $\omega(G)$ is the size of the largest complete subgraph in G.

Corollary 4.1. *If G has a representation modulo n, and p is a prime divisor of n then $p \geq \omega(G)$.*

We restate Lemma 2.10 and Corollary 2.12 from Evans, Isaak, and Narayan [9].

Lemma 5. *If G contains a $K_m + K_1$ as an induced subgraph and G is representable modulo n, then n contains at least m distinct prime factors.*

Corollary 5.1. *If G contains a $K_m + K_1$ and p_i is the smallest prime satisfying $p_i \geq \chi(G)$ then $rep(G) \geq p_i p_{i+1} \cdots p_{i+m-1}$, where $p_{i+1}, p_{i+2}, \ldots, p_{i+m-1}$ are the next $m-1$ primes larger than p_i.*

4 Complete Graphs minus disjoint paths

In this section we start by finding the representation number of $K_n - mP_{2j+1}$ (complete graph minus disjoint copies of paths of odd length). Our strategy will be to find a modular representation for $K_n - mP_{2j+1}$ by first finding its product (or coordinate) representation, thereby finding its Prague dimension and representation number.

4.1 Representation number of $K_n - mP_{2j+1}$

Theorem 6. *For $n \geq 3$ and $j \geq 1$, let $1 \leq m \leq \left\lfloor \frac{n}{2j+1} \right\rfloor$, then $rep(K_n - mP_{2j+1}) = p_s p_{s+1}$, where p_s is the smallest prime greater than or equal to $n - mj$ and p_{s+1} is the next highest prime after p_s.*

Proof. On removing m disjoint copies of P_{2j+1} from the complete graph K_n, the largest complete subgraph left in $K_n - mP_{2j+1}$ is K_{n-mj}. From Corollary 4.1 it follows that, $p_s \geq n - mj$. Moreover $K_n - mP_{2j+1}$ contains a $K_2 + K_1$, so from Corollary 5.1 we get, $rep(K_n - mP_{2j+1}) \geq p_s p_{s+1}$.

Next we show that $K_n - mP_{2j+1}$ has a representation modulo $p_s q$, where $q \geq p_{s+1}$. We give a coordinate representation with respect to mod p_s and mod q to the vertices of $K_n - mP_{2j+1}$ as follows. Let $v_1, v_2, \ldots, v_{2j+1}$ be the vertices of the "first removed path". Assign coördinates to these vertices as follows (the first coordinate is $\equiv$ mod p_s and the second coordinate is $\equiv$ mod q): for $1 \le i \le 2j+1$,

$$\text{if } i \text{ is odd, } v_i \quad \text{is assigned } \left(\left\lfloor\frac{i-1}{2}\right\rfloor, \left\lfloor\frac{i-1}{2}\right\rfloor\right),$$

$$\text{if } i \text{ is even, } v_i \quad \text{is assigned } \left(\left\lfloor\frac{i-1}{2}\right\rfloor, \left\lfloor\frac{i}{2}\right\rfloor\right)$$

Let $v_{2j+2}, v_{2j+3}, \ldots, v_{4j+2}$ be the vertices of the "second removed path". Assign coordinates to these vertices as follows: for $1 \le i \le 2j+1$,

$$\text{if } i \text{ is odd, } v_{2j+1+i} \quad \text{is assigned } \left((j+1) + \left\lfloor\frac{i-1}{2}\right\rfloor, (j+1) + \left\lfloor\frac{i-1}{2}\right\rfloor\right),$$

$$\text{if } i \text{ is even, } v_{2j+1+i} \quad \text{is assigned } \left((j+1) + \left\lfloor\frac{i-1}{2}\right\rfloor, (j+1) + \left\lfloor\frac{i}{2}\right\rfloor\right)$$

Proceeding along the same lines, let $v_{(r-1)(2j+1)+1}, v_{(r-1)(2j+1)+2}, \ldots, v_{r(2j+1)}$ be the vertices of the "r^{th} removed path", where $1 \le r \le m$. Assign coordinates to these vertices as follows: for $1 \le i \le 2j+1$,

$$\text{if } i \text{ is odd, } v_{(r-1)(2j+1)+i} \quad \text{is assigned } \left((r-1)(j+1) + \left\lfloor\frac{i-1}{2}\right\rfloor, (r-1)(j+1) + \left\lfloor\frac{i-1}{2}\right\rfloor\right),$$

$$\text{if } i \text{ is even, } v_{(r-1)(2j+1)+i} \quad \text{is assigned } \left((r-1)(j+1) + \left\lfloor\frac{i-1}{2}\right\rfloor, (r-1)(j+1) + \left\lfloor\frac{i}{2}\right\rfloor\right)$$

This is best illustrated in the following table:

vertices	mod p_s	mod q
v_1	0	0
v_2	0	1
$\vdots$	$\vdots$	$\vdots$
v_{2j+1}	j	j
v_{2j+2}	$j+1$	$j+1$
v_{2j+3}	$j+1$	$j+2$
$\vdots$	$\vdots$	$\vdots$
v_{4j+2}	$2j+1$	$2j+1$
$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$
$v_{m(2j+1)}$	$m(j+1)-1$	$m(j+1)-1$

Now we assign labels to the remaining $n - m(2j+1)$ vertices which do not lie on the paths removed. Let $v_{m(2j+1)+1}, v_{m(2j+1)+2}, \ldots, v_n$ be the vertices that do not lie on the paths removed. For $1 \leq i \leq n - m(2j+1)$, the vertex $v_{m(2j+1)+i}$ is assigned the label

$$\Big(mj + m + i - 1,\ mj + m + i - 1\Big).$$

We claim that this assignment of labels gives the coordinate representation of $K_n - mP_{2j+1}$ modulo $p_s q$. Note that the adjacency conditions follow from the remarks we made in Section 2. We just need to verify that we have sufficient number of labels. We focus on the smaller prime p_s. The number of residues modulo p_s that are consumed (first column of the table) in assigning labels to the vertices $v_1, \ldots, v_{m(2j+1)}$ is $m(j+1)$. The number of labels consumed (hence the number of residues) for the vertices $v_{m(2j+1)+1}, v_{m(2j+1)+2}, \ldots, v_n$ is $n - m(2j+1)$. Thus the total number of labels we require for this assignment is $n - mj$. As proved earlier, $p_s \geq n - mj$. Hence we will have sufficiently many residues modulo p_s to achieve the coordinate representation of $K_n - mP_{2j+1}$. ■

Corollary 6.1. *The Prague dimension of $K_n - mP_{2j+1}$ is* 2.

The following corollary follows immediately from Theorem 3.

Corollary 6.2. *$K_n - mP_{2j+1}$ does not contain either $K_2 + 2K_1$, or $K_3 + K_1$, or the complement of a chordless odd cycle of length at least* 5, *as an induced graph.*

4.2 Representation number of $K_n - mP_{2j}$

In this section we determine the representation number and the Prague dimension of $K_n - mP_{2j}$ (complete graph minus disjoint copies of paths of even length). But before we do that, we would like to draw reader's attention to a potential problem caused by *twin primes* in the case of even paths.

Example 4.1. Let $G = K_{20} - 5P_4$ and let p be a prime divisor of $rep(G)$. The largest complete subgraph in G is K_{10}. From Corollaries 4.1 and 5.1 it follows that $p \geq 10$ and $rep(G) \geq (11)(13)$ (note that $p = 11$ and $q = p + 2 = 13$ are *twin primes*). We will show that $rep(G) \neq (11)(13)$.

Suppose we construct a coordinate representation of G with respect to $\bmod 11$ and $\bmod 13$, where each vertex v_i is assigned a coordinate representation $(x_i \bmod 11, y_i \bmod 13)$. Let v_1, v_2, v_3, v_4 be the first path removed. The number of residues $\bmod 11$ we consume to assign x_i for $1 \leq i \leq 4$ (irrespective of the strategy of assigning labels) is at least 2 or 3, depending on which prime we "lean on". Likewise, the number of residues mod 13 we consume to assign y_i for $1 \le i \le 4$ is at least 3 or 2 respectively. In all, we consume a total of 5 residues (adding the number of residues used modulo both the primes) to assign labels for the first removed path. Similarly we will consume 5 total residues for each of the five paths removed. This requires a total of at least 25 residues, but with mod 11 and mod 13 we can only have a total of 24 residues. Thus a representation is not possible mod $(11 \cdot 13)$.

Remark. The purpose of this example is multi-fold: it shows that the possible appearance of twin primes makes the even path case more challenging and interesting as $rep(K_n - mP_{2j})$ need not be $p_s p_{s+1}$, it also shows that knowing the Prague dimension of a graph may not be sufficient to determine its representation number. Moreover it emphasizes the need for a more elaborate criteria for determining q than the one we developed for the odd path length case in Theorem 6.

Our first result is about the Prague dimension of $K_n - mP_{2j}$. We would like to point out that a more straightforward proof can be given for Theorem 7 instead of the construction that we have provided here. However our reason for using this construction early on is to establish the groundwork that is required for the proof of Theorem 9.

Theorem 7. *For $n \ge 3$ and $j > 1$, let $1 \le m \le \left\lfloor \frac{n}{2j} \right\rfloor$, p_s be the smallest prime greater than or equal to $n - mj$ and q_s be the smallest prime such that $q_s > p_s$ and $p_s + q_s \ge 2(n - mj) + m$. Then $K_n - mP_{2j}$ has a representation modulo $p_s q_s$, hence it has Prague dimension 2.*

Proof. Since $K_n - mP_{2j}$ contains $K_2 + K_1$, therefore it follows from Lemma 5 that $\dim_P(K_n - mP_{2j}) \ge 2$. Let p_s and q_s be the primes as described in the statement of the the theorem. We give a coordinate representation with respect to mod p_s and mod q_s as follows:

Let $v_1, v_2, \ldots, v_{2j}, \ldots, v_{(q_s - p_s - 1)2j+1}, \ldots, v_{(q_s - p_s)2j}$ be the vertices of the first $(q_s - p_s)$ removed paths. The coordinates to these vertices are assigned along the same lines as mentioned in the proof of Theorem 6 but with some modifications.

In this **first stage**, we "lean" on the higher prime q_s by consuming its residues more in comparison to that of the smaller prime p_s. We do this for the first $q_s - p_s$ paths removed. The assignment of labels is best explained in the following table.

vertices	mod p_s	mod q_s
v_1	0	0
v_2	0	1
$\vdots$	$\vdots$	$\vdots$
v_{2j}	$j-1$	j
v_{2j+1}	j	$j+1$
v_{2j+2}	j	$j+2$
$\vdots$	$\vdots$	$\vdots$
v_{4j}	$2j-1$	$2j+1$
$\vdots$	$\vdots$	$\vdots$
$v_{(q_s-p_s-1)2j+1}$	$(q_s-p_s-1)j$	$(q_s-p_s-1)(j+1)$
$v_{(q_s-p_s-1)2j+2}$	$(q_s-p_s-1)j$	$(q_s-p_s-1)(j+1)+1$
$\vdots$	$\vdots$	$\vdots$
$v_{(q_s-p_s)2j}$	$(q_s-p_s)j-1$	$(q_s-p_s)j+(q_s-p_s-1)$

If $m \geq q_s - p_s$ then at the end of this first stage a crucial point to be noted is that the number of residues left (to be consumed) mod p_s and the number of residues left (to be consumed) mod q_s are equal.

We proceed to the **second stage** of the label assignment if $m > q_s - p_s$; otherwise skip to the third stage. If $m > q_s - p_s$ (i.e. if there are vertices from the removed paths still left unlabeled after stage 1), we switch our strategy of leaning only on q_s to "alternately leaning" on p_s and q_s. So for the $(q_s - p_s + 1)^{st}$ path removed, we "lean on the prime p_s" and assign the coordinates as follows:

vertices	mod p_s	mod q_s
$v_{(q_s-p_s)2j+1}$	$(q_s-p_s)j$	$(q_s-p_s)(j+1)$
$v_{(q_s-p_s)2j+2}$	$(q_s-p_s)j+1$	$(q_s-p_s)(j+1)$
$\vdots$	$\vdots$	$\vdots$
$v_{(q_s-p_s+1)2j}$	$(q_s-p_s+1)j$	$(q_s-p_s+1)j+(q_s-p_s-1)$

In essence, to get the labels for the odd row of the $(q_s - p_s + 1)^{st}$ block we add $(j, j+1)$ to the corresponding odd rows of the $(q_s - p_s)^{th}$ block and to get the even rows of the $(q_s - p_s + 1)^{st}$ block, we add $(j+1, j)$ to the corresponding even rows of the $(q_s - p_s)^{th}$ block.

For the next path removed (i.e. the $(q_s - p_s + 2)^{nd}$ path) we "lean on the prime q_s" and assign the coordinates as follows: to get the coordinates for the odd row of the $(q_s-p_s+2)^{\text{nd}}$ block we add $(j+1, j)$ to the corresponding odd rows of the $(q_s - p_s + 1)^{\text{st}}$ block and to get the even rows of the $(q_s - p_s + 2)^{\text{nd}}$ block, we add $(j, j+1)$ to the corresponding even rows of the $(q_s - p_s + 1)^{\text{st}}$ block. This yields the following:

vertices	mod p_s	mod q_s
$v_{(q_s-p_s+1)2j+1}$	$(q_s - p_s + 1)j + 1$	$(q_s - p_s + 1)j + (q_s - p_s)$
$v_{(q_s-p_s+1)2j+2}$	$(q_s - p_s + 1)j + 1$	$(q_s - p_s + 1)(j + 1)$
$\vdots$	$\vdots$	$\vdots$
$v_{(q_s-p_s+2)2j}$	$(q_s - p_s + 2)j$	$(q_s - p_s + 2)j + (q_s - p_s)$

The idea is to alternate these two types of blocks until all remaining blocks are exhausted. For the purpose of an explicit procedure consider the following: To label the u^{th} row of the $(q_s - p_s + t)^{\text{th}}$ block, where $1 \leq t \leq m - (q_s - p_s)$ and $1 \leq u \leq 2j$, we add $\Big(j + [(t-u) \mod 2], j + 1 - [(t-u) \mod 2]\Big)$ to the u^{th} row of the $(q_s - p_s + t - 1)^{\text{th}}$ block. A key point here is that, with this procedure, the total number of residues (number of residues in both columns) that are consumed in labeling vertices of the each of the remaining $m - (q_s - p_s)$ paths is still $2j + 1$, same as in the first stage. The advantage we now have is that we will have sufficient number of residues left in both $\mod p_s$ and $\mod q_s$.

Now comes the **third stage** where we label the remaining $n - 2mj$ vertices which do not lie on the paths removed. To each vertex v_{2mj+i}, where $1 \leq i \leq n - 2mj$ we assign coordinates (x_i, y_i) such that each x_i is a distinct residue from the set of residues mod p_s left after stages 1 and 2 are complete, likewise each y_i is a distinct residue from the set of residues mod q_s left after the completion of stages 1 and 2.

We claim that this assignment of labels gives the coordinate representation of $K_n - mP_{2j}$ modulo $p_s q_s$. Note that the adjacency conditions follow from the remarks we made in Section 2. We just need to verify that we have sufficient number of labels.

Case (i). If $m \leq q_s - p_s$, then we will not have the second stage, in which case we directly go from the first to the third stage. In the first stage, we need mj residues $\mod p_s$ and in the third stage we need $n - 2mj$ residues $\mod p_s$. In all, we need at least $n - mj$ residues $\mod p_s$. Since $p_s \geq n - mj$, we will have sufficiently many residues to carry out the labeling assignment.

Case (ii). If $m > q_s - p_s$, then we will have all the three stages. We compute the total number of residues consumed modulo the prime p_s.

1. For the first stage, we consume $(q_s - p_s)j$ residues.
2. For the second stage, we consume $\lceil (j + \frac{1}{2})(m - q_s + p_s) \rceil$ residues.
3. For the third stage, we consume $n - 2mj$ residues.

In all, we need $(q_s - p_s)j + \lceil (j + \frac{1}{2})(m - q_s + p_s) \rceil + n - 2mj$ residues. We will have sufficiently many residues provided,

$$\begin{aligned} p_s &\geq (q_s - p_s)j + \left\lceil \left(j + \frac{1}{2}\right)(m - q_s + p_s) \right\rceil + n - 2mj \\ &\geq (q_s - p_s)j + \left(j + \frac{1}{2}\right)(m - q_s + p_s) + n - 2mj \\ p_s + q_s &\geq 2(n - mj) + m. \end{aligned}$$

But this is exactly the condition we have in the statement of the theorem. Hence we have a coordinate representation of $K_n - mP_{2j}$ modulo $p_s q_s$ and $\dim_P(K_n - mP_{2j}) = 2$. ■

The following corollary follows immediately from Theorem 3.

Corollary 7.1. *$K_n - mP_{2j}$ does not contain either $K_2 + 2K_1$, or $K_3 + K_1$, or the complement of a chordless odd cycle of length at least 5, as an induced graph.*

The following example highlights yet another challenge in finding $rep\,(K_n - mP_{2j})$. It shows that that the choice of the smallest prime p_s we made in Theorem 7 need not always give us the representation number.

Example 4.2. Let $G = K_{128} - 19P_6$, then $n - mj = 71$. According to the choice of primes we made in Theorem 7, we have $p_s = 71$ and $q_s = 97$ (thus $p_s + q_s \geq 2(n - mj) + m = 161$). Is $rep\,(K_{128} - 19P_6) = p_s q_s = 6887$? If we follow the construction in the proof of Theorem 7, then $p = 73$ and $q = 89$ also satisfy the condition $p + q \geq 2(n - mj) + m$. Therefore $K_{128} - 19P_6$ has a representation modulo $pq = 6497 < p_s q_s$. This means the choice of the prime p_s need not always be the optimal choice.

The following number-theoretic result is useful in finding the $rep\,(K_n - mP_{2j})$ in certain cases and also in establishing that it cannot be a product of three primes.

Lemma 8. *For $n \geq 3$ and $j > 1$, let $1 \leq m \leq \left\lfloor \frac{n}{2j} \right\rfloor$. Suppose there exists a prime p_s such that $n - mj \leq p_s < n - mj + m$ and let q_s be the smallest prime such that $q_s > p_s$ and $p_s + q_s \geq 2(n - mj) + m$. For any primes p_r, p_t such that $p_s < p_r \leq p_t < q_s$, the product $p_r p_t > q_s$.*

Proof. Since $n - mj + m > 1$, from Bertrand's postulate [17] it follows that there exists a prime $\hat{q}$ such that $\hat{q} \in [n-mj+m, 2(n-mj+m)] \subsetneq [n-mj+m, 2n]$. Observe that $\hat{q}$ satisfies $p_s + \hat{q} \geq 2(n-mj)+m$, consequently $q_s \leq \hat{q}$. From the bounds on m, it follows immediately that $\frac{n}{2} \leq n - mj \leq p_s$. Thus we have

$$\frac{n}{2} \leq p_s < q_s \leq \hat{q} < 2n. \tag{1}$$

Suppose $p_s < p_r \leq p_t \leq q_s$ and $p_r p_t \leq q_s$, then from inequality (1) we have $\frac{n^2}{4} < p_r p_t \leq 2n$, which implies $n < 8$. It can be easily verified that among the finitely many possibilities for the primes p_s, p_r, p_t and q_s that exist for $n < 8$ there are no cases for which $p_r p_t \leq q_s$. ∎

For the purpose of clarity we state the following corollary which is an easy consequence of Lemma 8.

Corollary 8.1. *Let p_s and q_s be the primes described in the lemma. The product of any three primes a, b and c such that $p_s < a \leq b \leq c < q_s$ will always be greater than $p_s q_s$.*

The next result gives the exact $rep(K_n - mP_{2j})$ in certain cases and bounds for it in other cases.

Theorem 9. *For $n \geq 3$ and $j > 1$, let $1 \leq m \leq \left\lfloor \frac{n}{2j} \right\rfloor$, p_s be the smallest prime greater than or equal to $n - mj$ and q_s be the smallest prime such that $q_s > p_s$ and $p_s + q_s \geq 2(n - mj) + m$.*

1. *If $p_s \geq n - mj + m$, then $rep(K_n - mP_{2j}) = p_s p_{s+1}$, where p_{s+1} is the next higher prime after p_s.*
2. *If $p_s < n - mj + m$, then $rep(K_n - mP_{2j}) \in [p_s p_{s+1}, p_s q_s]$ and it will be a product of two distinct primes both of which will lie in $[p_s, q_s]$.*

Proof. For the sake of a complete argument, we mention that for $j = 1$, the graph is $K_n - mP_2$, which is a complete multipartite graph and its representation number has already been discussed by Evans, Isaak and Narayan [9] (see Theorem 5.5 and Corollary 5.6). We proceed with our proof as follows:

On removing m disjoint copies of P_{2j} from the complete graph K_n, the largest complete subgraph left in $K_n - mP_{2j}$ is K_{n-mj}. Let z be the smallest prime divisor of $rep(K_n - mP_{2j})$. From Corollary 4.1 it follows that $n - mj \leq p_s \leq z$. Moreover $K_n - mP_{2j}$ contains a $K_2 + K_1$, so from Corollary 5.1 it follows that

$$p_s p_{s+1} \leq rep(K_n - mP_{2j}), \tag{2}$$

where p_{s+1} is the next highest prime after p_s.

From the proof of Theorem 7 we have that $K_n - mP_{2j}$ has a representation modulo $p_s q_s$. Now we will show that the condition given on $p_s + q_s$ in the statement of the theorem is necessary. Let $v_1, v_2, \ldots, v_{2j}$ be the vertices of the "first removed path". Irrespective of the strategy we use to assign the coordinates $(x_i \pmod{p_s}, y_i \pmod{q_s})$ to each vertex v_i, where $1 \leq i \leq 2j$, the total number of residues (adding the number of residues we consume for both x_i and y_i) we consume is at least $2j+1$. This holds for all the m paths that we remove. Hence any strategy will consume at least $m(2j+1)$ total residues to label the vertices that are removed. For the remaining $n - 2mj$ vertices (the ones which do not lie on the paths removed), with the adjacency requirements, we will need at least $n - 2mj$ residues with respect to each of the primes p_s and q_s, hence a total of $2(n-2mj)$ residues. In all, the total number of residues we need is at least $2(n - mj) + m$, thus $p_s + q_s \geq 2(n - mj) + m$. Consequently we have

$$rep(K_n - mP_{2j}) \leq p_s q_s. \tag{3}$$

Now we consider the two cases:

Case 1: Suppose $p_s \geq n - mj + m$, then the next highest prime $p_{s+1} > n - mj + m$, consequently $p_s + p_{s+1} \geq 2(n - mj) + m$. This implies p_{s+1} satisfies all the conditions on q_s and $q_s = p_{s+1}$. From (2) and (3) it follows that $rep(K_n - mP_{2j}) = p_s p_{s+1}$.

Case 2: Suppose $n - mj \leq p_s < n - mj + m$. From (2) and (3) it follows that $p_s p_{s+1} \leq rep(K_n - mP_{2j}) \leq p_s q_s$. Using Corollary 8.1 we can easily rule out the possibility of the representation number being a product of three primes, said differently, $rep(K_n - mP_{2j})$ will be a product of two distinct primes. Next we can use Lemma 8 and the remark we made at the start of the proof about the smallest prime divisor of $rep(K_n - mP_{2j})$ to establish that the two prime divisors of $rep(K_n - mP_{2j})$ will lie in the interval $[p_s, q_s]$.

■

We conclude by suggesting that the methods we have developed here can be extended to the graphs of the form $K_n - mP_{2j+1} - lP_{2j}$ and also to other complete graphs minus multiple paths of distinct odd and even lengths. It might be interesting to explore complements of other graphs. The family of G minus a disjoint union of stars was investigated by Agarwal, Lopez, and Narayan [1]. While it would surely be a challenge to explore $rep(G - T_n)$ and $\dim_P(G - T_n)$ where T_n is a tree on n vertices, it is likely that some results could be obtained for certain classes of trees. Another problem of interest could be to get the exact representation number of $K_n - mP_{2j}$ when $n - mj \leq p_s < n - mj + m$ by closing the bounds we have in (2) and (3).

Acknowledgements: The authors would like to thank the referee for their careful reading and valuable advice which resulted in the corrected version of Theorem 9 and a significant improvement of this article.

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