Cyclic Loop Designs

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ABSTRACT. We give cyclic constructions for loop designs with block size $k = 3, 4$, and 5, and all values of v, and we thereby determine the (v, λ) spectrum for LDs with these block sizes. For $k = 3, 5$ the (v, λ) spectrum for LDs is the same as that for cyclic LDs, but this is not true for $k = 4$.

1. Introduction

The purpose of this note is to give cyclic constructions for a new type of combinatorial design or graph design which we call a loop design. We use the notation LD(v, k, λ, j). From the graph point of view, we decompose $K_v(\lambda, j)$, the complete (multi)graph on v points with λ multiple edges for each pair of points and with j loops at each vertex, into ordered blocks $(a_1, a_2, ..., a_{k-1}, a_1)$ of size k. Each block is the subgraph which contains the unordered edges $\{a_s, a_j\}$, for each pair of consecutive edges in the list, and which contains the loop at vertex a_1. The block (a, b, c, d, a) contains the unordered edges $\{a, b\}$, $\{b, c\}$, $\{c, d\}$, $\{d, a\}$ and the loop $\{a, a\}$. Each block consists of two cycles (or loops) of lengths 1 and $k - 1$ which share a common vertex. When convenient, we denote blocks simply as *aba* or *abca*, etc.

These designs were introduced in [5], and there the necessary conditions were shown to be sufficient for the existence of LDs for $k = 3, 4$, or 5. Here we show that the same necessary conditions (Lemma 1, below) aid us in the determination of the (λ, v) spectrum for the existence of cyclic LDs.

It may be noted that LDs have characteristics of other designs such as Mendelsohn designs, which apply the idea of cyclic triples of ordered pairs, and balanced ternary designs (BTDs) in which a point appears in a block 0, 1 or 2 times [4], or cycle designs in which a graph is decomposed into copies of C_k. Undefined terms can be found in [2] or [6].

A balanced incomplete block design, a BIBD(v, k, λ), is a decomposition of K_v into subsets of size k (viewed as copies of K_k, the complete graph on k vertices). BIBDs play an important role in later sections, and we give the well-known necessary conditions for existence of BIBDs: $vr = bk$ and $\lambda(v - 1) = r(k - 1)$. Here b is the number of blocks and r is the replication number (the number of blocks in which each point appears).

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We will define j_i to be the number of blocks in which, say, point x occurs as an interior point in an LD block. This means there are $2j$ non-loop edges incident with x in blocks in which x is the endpoint and $2j_i$ edges incident with x in those blocks in which x is an interior point. It follows that $2j + 2j_i = \lambda(v-1)$, and this equation is independent of x. This shows that j_i is a constant for any point in an LD, and, therefore, it is proper to define a replication number for LDs by $r = j + j_i = \lambda(v-1)/2$.

The equations just derived also imply that $\lambda(v-1)$ is necessarily even, but this fact is also a consequence of the equation in part (a) Lemma 1, below, which gives the fundamental necessary condition.

LEMMA 1. [5] *For any $LD(v, k, \lambda, j)$, it is necessary that*

THEOREM 1. *(a)* $j = \frac{\lambda(v-1)}{2(k-1)}$; *and*
(b) $j_i = j(k-2)$.

PROOF. From the definition of LD, the number of blocks and the number of loops is the same. Part (a) follows from $\frac{\lambda}{k-1}\binom{v}{2} = jv$. The right hand side counts the number of loops (blocks), and the left hand side is the number of non-loop edges divided by the number of such edges per block. Part (b) follows from part (a) and the discussion of j_i just above. □

In the next three sections we show that the (v, λ) spectrum for cyclic LDs is the same as that for LDs provided $k = 3, 5$. When $k = 4$, however, the situation will be quite different.

2. Cyclic Loop Designs With $k = 3$.

When $k = 3$, an LD$(v, 3, \lambda, j)$ satisfies $j = \lambda(v-1)/4$, by Lemma 1. Since each block has the form (a, b, a), each block represents a 2-cycle in which one vertex has a loop. Moreover, it follows that λ must be even.

THEOREM 2. *There exists a cyclic $LD(2t+1, 3, 2, t)$ for every $t \geq 1$. There exists a cyclic $LD(2t, 3, 4, 2t-1)$ for every $t \geq 2$.*

PROOF. If $v = 2t+1$, the starter blocks are $0i0$ for $1 \leq i \leq t$. When v is even, the minimum λ is 4. An LD$(2t, 3, 4, 2t-1)$ may be constructed by including starter blocks $0i0$ and $i0i$ for $1 \leq i \leq t$. □

THEOREM 3. *The necessary conditions for existence of LDs with $k = 3$ are sufficient for the existence of cyclic $LD(v, 3, \lambda, j)$.*

PROOF. By Lemma 1, λ and j are directly proportional. Higher values $\lambda = 2s$ (for odd v) and $\lambda = 4s$ (even v) may be obtained usimg multiple copies of the base design (and multiplying j by s) in the previous theorem. □

3. Cyclic Loop Designs With $k = 4$.

Lemma 1 implies that $j = \lambda(v-1)/6$ is a necessary condition when $k = 4$. It is convenient to divide the discussion into the different cases for v mod 6 because of the following lemma [5] which gives the minimal index λ in each case.

LEMMA 2. *A loop design with parameters $(v, 4, \lambda, j)$ exists if and only if (λ, j) is an integer multiple of the minimal values given in the table:*

v	$6t$	$6t+1$	$6t+2$	$6t+3$	$6t+4$	$6t+5$
(λ, j)	$(6, 6t-1)$	$(1, t)$	$(6, 6t+1)$	$(3, 3t+1)$	$(2, 2t+1)$	$(3, 3t+2)$

For each positive integer $v \geq 7$ we define a Difference Triple (see p.32 of [6]), or DT, to be a set of three positive integers (x, y, z) such that (1) x, y, and z are distinct and in the set $\{1, 2, ..., v-1\}$; (2) $x+y \equiv z \pmod{v}$ or $x+y+z \equiv 0 \pmod{v}$. A DT determines a base block for a cyclic triple system $(x, y, z) \rightarrow \{0, x, x+y\}$. We in turn use the DT to create a base block $(0, x, x+y, 0)$ for a loop design.

A base block $\{a, b, c\}$ in a cyclic design has a full orbit if it is used to generate the set of blocks $\{a+i, b+i, c+i\}$ for $0 \leq i \leq v-1$. Any base block may be transformed into a base block for a cyclic LD under what we call the natural map:

$$\{a, b, c\} \rightarrow (a, b, c, a).$$

LEMMA 3. *Suppose a BIBD$(v, 3, \lambda)$ is cyclically generated by j base blocks which have full orbits. Then the base blocks, under the natural map, determine blocks for a cyclic LD$(v, 4, \lambda, j)$. Conversely, if the j base blocks of a cyclic LD$(v, 4, \lambda, j)$ have full orbits, then the reverse of the natural map creates a set of base blocks for a cyclic BIBD$(v, 3, \lambda)$.*

PROOF. It is only necessary to observe that the corresponding blocks of the two designs determine the same triangle of regular edges and that each point in the loop design is an end-point of a block (has a loop) exactly j times. □

EXAMPLE 1. *A BIBD$(15, 3, 1)$ is cyclically generated by the base blocks $\{0, 1, 4\}$, $\{0, 2, 9\}$, and $\{0, 5, 10\}$. The third base block is described as a short block and is only used for one-third of the usual development: $\{0, 5, 10)$, $\{1, 6, 11\}$, $\{2, 7, 12\}$, $\{3, 8, 13\}$, and $\{4, 9, 14\}$. At this point, the blocks form a parallel class and each pair of points which differ by 5 or 10 already appears once. This set of blocks is not j-balanced, a requirement for an LD. For $v = 15$, the condition on j (Lemma 1) requires $\lambda = 3$.*

THEOREM 4. *There exist cyclic LD$(6t+1, 4, 1, t)$, cyclic LD$(6t+3, 4, 3, 3t+1)$, and cyclic LD$(6t+5, 4, 3, 3t+2)$ for all $t \geq 1$.*

PROOF. There are complete sets of difference triples for $v = 6t+1$ and $v = 6t+3$, except for $v = 9$ (see Ch. 7 of [3] or Appendix A of [6]). We use the known difference triples (all with full orbits) for $v = 6t+1$ to create cyclic LD$(6t+1, 4, 1, t)$. For $v = 6t+3$, use the each DT to create 3 base blocks $(0, x, x+y, 0)$, $(x, x+y, 0, x)$, and $(x+y, 0, x, x+y)$. These will create an LD$(6t+3, 4, 3, 3t+1)$, whether or not a base block is short - except for $v = 9$. For LD$(9, 4, 3, 4)$, a suitable set of base blocks is $(0, 2, 3, 0)$, $(0, 3, 2, 0)$, $(0, 5, 3, 0)$, and $(0, 4, 8, 0)$. We have proved that minimal index cyclic LD$(v, 4, \lambda, j)$ exist for $v = 6t+1$ or $v = 6t+3$. For $v = 6t+5$, use the full base blocks $\{0, i, 2i, 0\}$ for $1 \leq i \leq 3t+2$. □

Sets of difference triples with $k = 3$ and even v are in Tables in Chapter 7 of [3], and these give the smallest possible index for a cyclic BIBD. By Lemma 3, no index for any cyclic LD can be smaller than any developed as in the tables. Further, if a short block is used, then to balance j, we have to triple the index in the table. We summarize the results in the theorem below.

THEOREM 5. *There is a cyclic* $LD(2n, 4, \lambda, j)$ *with minimal index* λ *given in the table below (the * in the table indicates where short blocks were used for the BIBD).*

v	$12t$	$12t+2$	$12t+4$	$12t+6$	$12t+8$	$12t+10$
λ	6*	12	4	12*	6	4

For every possible value of v we have constructed a minimal index cyclic LD, and since for any LD (by Theorem 1) the index is neccessarily a multiple of the minimal one, the existence problem for LDs with $k = 4$ is solved, and we have proved the following which may be contrasted with Lemma 2:

THEOREM 6. *The necessary conditions for the existence of* $LD(v, 4, \lambda, j)$ *are sufficient for the existence of cyclic* $LD(v, 4, \lambda, j)$ *if* v *is odd, if* $v = 6t$, *or if* $v = 12t+8$. *However, if* $v = 12t+2$, $12t+4$, *or* $12t+10$, *then cyclic* $LD(v, 5, 2\lambda, 2j)$ *exist if and only if non-cyclic* $LD(v, 4, \lambda, j)$ *exist.*

4. Cyclic Loop Designs With $k = 5$.

For convenience, we note that, in this section, $j = \lambda(v-1)/8$. We begin with several examples for small n.

EXAMPLE 2. *An example with* 12 *blocks, a cyclic* $LD(4, 5, 8, 3)$. *The columns are blocks, and the index is minimal. The first 3 columns are base blocks.*

1	1	1	2	2	2	3	3	3	4	4	4
2	2	3	3	3	4	4	4	1	1	1	2
3	4	2	4	1	3	1	2	4	2	3	1
4	3	4	1	4	1	2	1	2	3	2	3
1	1	1	2	2	2	3	3	3	4	4	4

Table 1: The blocks of an LD(4, 5, 8, 3).

EXAMPLE 3. *A cyclic* $LD(5, 5, 2, 1)$ *is generated by the starter block* $(0, 2, 3, 1, 0)$.

EXAMPLE 4. *An* $LD(6, 5, 8, 5)$. *The blocks are generated cyclically mod 5 using starter blocks* $\infty 210 \infty$, $1\infty 341$, $3\infty 123$, $0\infty 420$, 12341, 02410.

EXAMPLE 5. *An* $LD(7, 5, 4, 3)$. *The base blocks are* 03620, 01230, 01350.

EXAMPLE 6. *An* $LD(8, 5, 8, 7)$ *with starter blocks expanded mod seven:* 02140, 01420, 04210, 02140, $5\infty 635$, $6\infty 356$, $3\infty 563$, $\infty 356\infty$.

EXAMPLE 7. *A cyclic* $LD(9, 5, 1, 1)$ *is generated modulo* 9 *by the starter block* $(0, 2, 5, 4, 0)$. *This is the smallest possible example with index* $\lambda = 1$.

EXAMPLE 8. *An* $LD(10, 5, 8, 9)$ *with starter blocks expanded mod* 9*: use* 7 *copies of* $(0, 4, 5, 3, 0)$ *and one copy each of* $(\infty, 1, 0, 2, \infty)$, $(1, \infty, 2, 0, 1)$, $(4, \infty, 3, 0, 4)$, *and* $(3, \infty, 4, 0, 3)$.

EXAMPLE 9. *An* $LD(11, 5, 4, 5)$, *with minimal index* 4. *The design is cyclic, generated mod* 11 *by the starter blocks:* 04530, 04530, 05940, 05610, 02530.

In the rest of this section, we show the necessary conditions are sufficient for existence of cyclic loop designs with $k = 5$ and with $v = 24t + s$. For even v, the minimum index will be 8, and for odd v, the minimum index is 1, 2 or 4, as determined from Lemma 1(a).

We use the convenient notation $(a, b, c, d, a) \times n$ to mean "use n copies of the block (a, b, c, d, a)."

EXAMPLE 10. *An* $LD(12,5,8,11)$ *is generated mod* 11 *using the following blocks:* $(0,4,5,3,0)\times 8$, $(\infty,5,0,6,\infty)$ *and* $(5,\infty,6,0,5)\times 3$.

The only examples for $k=5$ and with index 1 occur for $v=8t+1$.

THEOREM 7. *There exist* $LD(v,5,1,j)$ *for* $v\equiv 1,9,17\ (mod\ 24)$.

PROOF. Suppose $v=8t+1$. Then there is a cyclic LD$(v,5,1,t)$ generated by the t starter blocks $(0,4s,8s-3,4s-1,0)$ whre $1\le s\le t$. □

The difference family given above has a striking property which was exploited for certain odd values of v in [5]. The differences between adjacent elements in $(0,4,5,3,0)$ are, respectively, left to right: $4,1,2,3$. When $s=2$, the differences are $8,5,6,7$, and so on. We exploit this regularity of the differences to obtain difference families for other values of v as well.

24t+2: For LD$(24t+2,5,8,j)$, use starter blocks, expanded mod $24t+1$. Use $(0,4s,8s-3,4s-1,0)\times 8$, for $1\le s\le 3t-1$. Use also $(0,12t-1,24t-3,12t-2,0)\times 2$, $(0,12t-2,24t-5,12t-3,0)\times 2$ and $(0,12t-3,24t-4,12t-1,0)\times 2$. Also use $(\infty,12t,0,12t+1,\infty)$, $(12t+1,\ \infty,12t,0,12t+1)\times 3$.

24t+3: For an LD$(24t+3,5,4,j)$ use the starter blocks $(0,4s,8s-3,4s-1,0)\times 4$ for $1\le s\le 3t-1$, and $(0,12t,24t-3,12t-1,0)\times 2$. Also, use one copy each of block $(0,12t-3,24t-2,12t+1,0)$, with differences $12t-3$ and $12t+1$ twice each; block $(0,12t+1,24t+1,12t,0)$ with differences $12t+1$ and $12t$ twice each; and block $(0,12t-2,24t-3,12t-1,0)$ with differences $12t-2$, $12t-1$ twice each.

24t+4: We have an LD$(24t+4,5,8,j)$ by expanding these starter blocks mod $24t+3$: $(0,4s,8s-3,4s-1,0)\times 8$ for $1\le s\le 3t$. Also use $(12t+1,\ \infty,12t+2,0,12t+1)\times 3$ and block $(\infty,12t+1,0,12t+2,\infty)$.

24t+5: We assume $v\ge 29$ and get LD$(24t+5,5,2,j)$. Use the starter blocks $(0,4s,8s-3,4s-1)\times 2$ for $1\le s\le 3t$, and use the block $(0,12t+1,24t+3,12t+2,0)$.

24t+6 Expand the following blocks mod $24t+5$: $(0,4s,8s-3,4s-1,0)\times 8$, for $1\le s\le 3t$. Use $(0,12t+2,24t+5,12t+1)\times 2$, and $(\infty,12t+2,0,12t+1,\infty)$, and $(12t+2,\ \infty,12t+1,0,12t+2)\times 3$.

24t+7: For LD(24t+7, 5, 4, j) use (0,4s,8s-3,4s-1,0)×4 for $1\le s\le 3t$. Also use (0, 12t+1, 24t+3, 12t+2,0), $(\infty,12t+4,0,12t+3,\infty)$, (12t+3,∞, $12+4,0,12t+3)$, (12t+1, $\infty,12t+2,0,12t+1)\times 2$.

24t+8 Expand mod $24t+7$. Use $(0,4s,8s-3,4s-1,0)\times 8$ for $1\le s\le 4t$. Use $(0,12t+4,24t+5,12t+3,0)\times 6$. Use $(\infty,12t+4,0,12t+1)$, $(12t+4,\ \infty,12t+1,0,12t+4)$. Lastly use two copies of $(12t+2,\ \infty,12t+3,0,12t+2)$.

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24t+10 Expand mod $24t+9$: $(0,4s,8s-3,4s-1,0)\times 8$ for $1\le s\le 3t$. Use $(12t+4,\infty,12t+1,0,12t+4)$, $(\infty,12t+4,0,12t+1,\infty)$, $(12t+2,\infty,12t+3,0,12t+2)\times 2$.

24t+11 There exist LD$(24t+11,5,4,j)$ for $t\ge 0$.Use blocks $(0,4s,8s-3,4s-1,0)\times 4$ for $1\le s\le 3t$. Use $(0,12t+4,24t+5,12t+3,0)\times 2$. Finally, use the three blocks $(0,12t+5,24t+9,12t+4,0)$, $(0,12t+5,24t+6,12t+1,0)$, and $(0,12t+2,24t+5,12t+3,0)$.

24t+12 There exist LD$(24t+12,5,8,j)$ for $0\le t$. Use $(0,4s,8s-3,0)\times 8$, for $1\le s\le 3t+1$. Use also $(\infty,12t+5,0,12+6,\infty)$ and $(12t+5,\infty,12t+6,0,12t+5)\times 3$.

24t+13 There exist cyclic $LD(24t+13,5,2,j)$. Use base blocks $(0,4s,8s-3,4s-1,0)\times 2$ for $1\le s\le 3t+1$, and use $(0,12t+5,12t+11,12t+6,0)$.

24t+14 Expand these blocks mod $24t+14$: $(0,4s,8s-3,4s-1,0)\times 8$, for $1\le s\le 3t+1$. Use $(\infty,12t+1,0,12t+5)$ and $(12t+5,\infty,12t+6,0,12t+5)\times 3$.

24t+15 There exists LD$(24t+15,5,4,j)$ for $s\ge 0)$.Use the starter blocks $(0,4s,8s-3,4s-1,0)\times 4$, for $1\le s\le 3t+1$. Use $(0,12t+8,24t+13,12t+7,0)\times 2$. In this last block, the differences $12t$+7 and $12t$+8 both occur, but they are additive inverses mod v. The other two differences mod v in this block (which we use twice) are $12t+5$ and $12t+6$. It thus only remains to use $12t+5$ and $12t+6$ as differences twice more each, and we do this with block $(0,12t+5,24t+11,12t+6,0)$.

24t+16 Expand mod $24t+15$: Use $(0,4s,8s-3,4s-1,0)\times 8$ for $1\le s\le 3t+1$. Use $(0,12t+5,24t+11,12t+6,0)\times 4$. Use $(\infty,12t+7,012t+8,\infty)$ and $(12t+7,\infty,12t+8,0,12t+7)\times 3$.

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24t+18 Expand mod $24t+17$. Use $(0,4s,8s-3,4s-1,0)\times 8$ for $1\le s\le 3t+1$. Use $(0,12t+6,24t+13,12t+7,0)\times 6$. Use $(\infty,12t+9,0,12t+10,\infty)$, and $(12t+9,\infty,12t+10,0,12t+9)\times 3$.

24t+19 Use base blocks $(0,4s,8s-3,4s-1,0)\times 4$ for $1\le s\le 3t+2$.*Also* use $(\infty,12t+9,0,12t+10,)$ and $(12t+9,,12t+10,0,12t+9)$.

24t+20 Expand mod $24t+19$. Use $(0,4s,8s-3,4s-1,0)\times 8$ for $1\le s\le 3t+2$. Use $(\infty,12t+9,0,12t+10,\infty)$, $(12t+9,\infty,12t+10,0,12t+9.)\times 3$.

24t+21 There exist LD$(v,5,2,j)$ for $v=24t+21$, $t\ge 0$.Use starter blocks $(0,4s,8s-3,4s-1,0)\times 2$ for $1\le s\le 3t+2$, and use $(0,12t+10,24t+19,12t+9,0)$.

24t+22 Expand mod $24t+21$. Use $(0,4s,8s-3,4s-1,0)\times 8$ for $1\le s\le 3t+2$. Use $(0,12t+9,24t+19,12t+10,0)\times 2$ and use $(\infty,12t+9,0,12t+10,\infty)$, $(12t+9,\infty,12t+10,0,12t+9)\times 3\times$.

24t+23 There exist LD$(v,5,4,j)$ for $v=24t+23$, $t\ge 0$. Use starter blocks $(0,4s,8s-3,4s-1,0)\times 4$ for $1\le s\le 3t+2$. Use $(0,12t+12,24t+21,12t+11,0)\times 2$ and one copy of $(0,12t+9,24t+19,12t+10,0)$.

24t Expand mod $24t+23$. Use $(0,4s,8s-3,4s,0)\times 8$ for $1\le s\le 3t+2$. Use $(0,12t+9,24t+19,12+10,0)\times 4$ Use $(\infty,12t+11,0,12t+12,\infty)$ and $(12t+11,\infty,12t+12,0,12t+11)\times 3$.

This shows LD$(v,5,\lambda,j)$ exist for every v and with minimal index. In this section, it is shown that the (v,λ) spectrum of cyclic LDs is the same as that of LDs which are not necessarily cyclic.

THEOREM 8. *The necessary conditions for existence of LDs is sufficient for the existence of cyclic LDs when $k=5$.*

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