Acyclic Kernel Number of Oriented Cycle Related Graphs

Indra Rajasingh, Bharati Rajan, S. Little Joice
Department of Mathematics, Loyola College,
Chennai 600 034, India.
littlejoice82@yahoo.com

Abstract

A kernel in a directed graph $D(V,E)$ is a set S of vertices of D such that no two vertices in S are adjacent and for every vertex u in $V \setminus S$ there is a vertex v in S, such that $(\overrightarrow{u,v})$ is an arc of D. The problem of existence of a kernel is NP-complete for a general digraph. In this paper we introduce the acyclic kernel problem for an undirected graph G and solve it in polynomial time for certain cycle related graphs.

Keywords: oriented graph, kernel, acyclic kernel number, NP-complete, topological ordering, ascent graph.

1 Introduction and Terminology

A kernel [7] in a directed graph $D(V,E)$ is a set S of vertices of D such that no two vertices in S are adjacent and for every vertex u in $V \setminus S$ there is a vertex v in S, such that $(\overrightarrow{u,v})$ is an arc of D. The concept of kernel in digraphs was introduced in different ways [12, 18].

Kernels arise naturally in the analysis of certain two-person positional games. Von Neumann and Morgenstern [18] were the first to introduce kernels when describing winning positions in 2 person games. They proved that any directed acyclic graph has a unique kernel. Not every digraph has a kernel and if a digraph has a kernel, this kernel is not necessarily unique. All odd length directed cycles and most tournaments have no kernels [2, 3].

If D is finite, the decision problem of the existence of a kernel is NP-complete for a general digraph [6, 17], and for a planar digraph with in-degrees ≤ 2, out-degrees ≤ 2 and degrees ≤ 3 [8]. It is further known that a finite digraph all of whose cycles have even length has a kernel [15], and that the question of the number of kernels is NP-complete even for this restricted class of digraphs [16].

The concept of kernel is widespread and appears in diverse fields such as logic, computational complexity, artificial intelligence, graph theory, game theory, combinatorics and coding theory [2, 3]. Efficient routing among a set of mobile hosts is one of the most important functions in ad hoc wireless networks. Dominating set based routing to networks with unidirectional links is proposed in [1, 11]. A new interest for these studies arose due to their applications in finite model theory. Indeed, variants of kernel are the best properties to provide counter examples of 0-1 laws in fragments of monadic second order logic [10].

In this paper we view the kernel problem from a different perspective. In the literature, only the existence of kernel of a digraph D and its applications are extensively studied [13]. Our aim in this paper is to investigate acyclic orientations of an undirected graph G and determine the acyclic kernel number of G.

2 Kernel in Oriented Graphs

An orientation of an undirected graph G is an assignment of exactly one direction to each of the edges of G. There are $2^{|E|}$ orientations for G. Let $O_x(G)$ denote the set of all orientations of G. For an orientation $O \in O_x$, let $G(O)$ denote the directed graph with orientation O and whose underlying graph is G.

An orientation O of an undirected graph G is said to be an acyclic orientation if it contains no directed cycles. Let $O_a(G)$ denote the set of all acyclic orientations of G.

Definition 1 *[7] A kernel in a directed graph $D(V, E)$ is a set S of vertices of D such that no two vertices in S are adjacent and for every vertex u in $V \setminus S$ there is a vertex v in S, such that $(\overrightarrow{u, v})$ is an arc of D. u is called the tail and v is called the head of the arc $(\overrightarrow{u, v})$.*

Definition 2 *Let $D(V, E)$ be any directed graph. The in-neighborhood of a vertex v, denoted by $N^-(v)$ is the set of all tail vertices with head vertex v. The out-neighborhood of a vertex v, denoted by $N^+(v)$ is the set of all head vertices with tail vertex v. $|N^+(v)|$ is called the out-degree of v and $|N^-(v)|$ is called the in-degree of v.*

Definition 3 *[13] The kernel number κ_x of G is defined as*

$$\kappa_x(G) = min\{\kappa(O) : O \in O_x(G)\}$$

where $\kappa(O) = min\{|K| : K$ is a kernel of $G(O)\}$.

Definition 4 *[13] The acyclic kernel number κ_a of G is defined as*

$$\kappa_a(G) = min\{\kappa(O) : O \in O_a(G)\}$$

where $\kappa(O) = min\ \{|K| : K$ *is a kernel of* $G(O)\}$.

The Acyclic Kernel Problem

Definition 5 *The acyclic kernel problem of an undirected graph* G *is to find a kernel* K *of* $G(O)$ *for some acyclic orientation* O *of* G *such that* $|K| = \kappa_a$.

3 Label Induced Kernel

A graph labeling is an assignment of integers to the vertices or edges or both, subject to certain conditions. Graph labeling has a wide range of applications. For instance, we can find labeling of graphs showing up in x-rays, crystallography, coding theory, radar, astronomy, circuit design and communication network addressing [4, 5]. Their theoretical applications are numerous, not only within the theory of graphs but also in other areas of mathematics such as combinatorial number theory, linear algebra and group theory admitting a given type of labeling [9].

Definition 6 *Let* G *be a graph with vertex set* $V(G)$ *and edge set* $E(G)$. *Then a labeling* f *of* G *is an injection from* $V(G)$ *to* $\{1, 2, ..., |V|\}$.

Definition 7 *[14] Let* G *be an undirected graph. A labeling* f *from the vertex set* $V(G)$ *to* $\{1, 2, ..., |V|\}$ *is said to induce a digraph* $G(f)$ *if* $E(G(f))$ *satisfies the following condition:* $(\overrightarrow{u, v}) \in E(G(f))$ *if and only if* $f(u) < f(v)$. *The labeling is called a topological ordering and the digraph is called an ascent graph.*

Theorem 1 *A digraph* G *is an ascent graph if and only if it is acyclic.*

Proof. Let G be an ascent graph. Then there exists an onto function $f : V(G) \rightarrow \{1, 2, ..., |V|\}$ such that $(\overrightarrow{u, v}) \in E(G)$ if and only if $f(u) < f(v)$. Suppose there exists a cycle $C : u_o u_1 ... u_k u_o$ in G. This implies $f(u_0) < f(u_k)$. But $(\overrightarrow{u_k, u_0}) \in E(C)$. Hence $f(u_k) < f(u_0)$, a contradiction. Thus G is acyclic. Conversely let G be an acyclic digraph. Then there exists at least one linear ordering $v_1 < ... < v_n$ of the vertex set $V(G) = \{v_1, v_2, ..., v_n\}$ such that an edge (v_i, v_j) of G has the direction $(\overrightarrow{v_i, v_j})$ if and only if $v_i < v_j$ [14]. Since G has topological ordering, G is an ascent graph. □

In the sequel we obtain a lower bound for the acyclic kernel number of certain cycle related graphs and prove that the lower bound is tight.

4 Kernel in Circular fan $F(m,k)$ with k Chords

Definition 8 *Let $C : x_1x_2...x_mx_1$ be a cycle on m vertices. Let u be a new vertex. The graph obtained by adding edges (u, x_i) $i = 1, 2, ..., m-2k$ to C and chords (x_m, x_{m-2}),*
(x_{m-2k+3}, x_{m-2k+1}) and $(x_{m-i}, x_{m-(i+3)})$, where $i = 1, 3, ..., 2k-5$ is called a circular fan with k chords and is denoted by $F(m,k)$, $k \geqslant 2$. The new edges are called spokes of $F(m,k)$. See Figure 1.

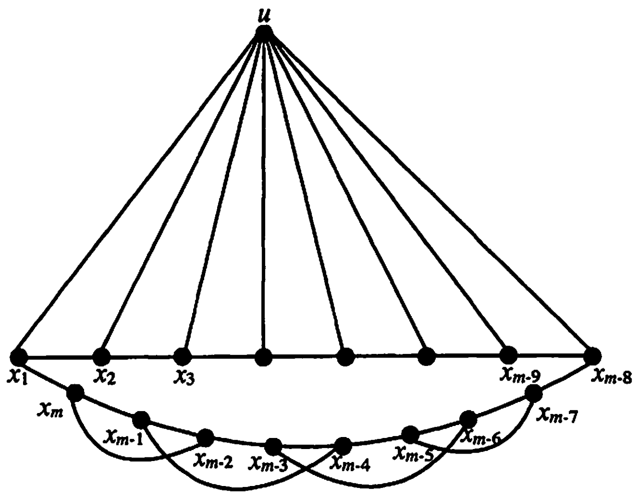

Figure 1: Circular fan with four chords $F(m,4)$

Theorem 2 *Let G be the circular fan $F(m,k)$with $k \geqslant 2$ chords and $m \geqslant 8$. Then $\kappa_a = \lceil \frac{k}{2} \rceil + 1$.*

Proof. Let G be the circular fan $F(m,k)$with $k \geqslant 2$ chords and $m \geqslant 8$.

Case 1: $\lceil \frac{k}{2} \rceil$ is even: Define a labeling $f : V(G) \to \{1, 2, ..., m, m+1\}$ by $f(u) = m+1$, $f(x_{m-(8i-6)}) = \{m, m-1, ..., m - \lceil \frac{k}{4} \rceil + 1\}$and $f(x_{m-(8i-3)}) = \{m - \lceil \frac{k}{4} \rceil, ..., m - \lceil \frac{k}{2} \rceil + 1\}$ where $i = 1, 2, ..., \lceil \frac{k}{4} \rceil$ Let the remaining vertices be labeled as $1, 2, ..., m - \lceil \frac{k}{2} \rceil$ arbitarily by f. Orient the edges $(\overrightarrow{v,w}) \in E(G)$ if and only if $f(v) < f(w)$, rendering f a topological ordering. Then by theorem 1, G is acyclic. We claim that vertices labeled

$m+1, m, m-1, ..., m-\lceil\frac{k}{2}\rceil+1$ constitute an acyclic kernel set K of G. Since u is a vertex of degree $m-2k$, u covers atmost $m-2k$ vertices of G. All the remaining $2k$ vertices of G are of degree 3. Thus at least $\lceil\frac{k}{2}\rceil+1$ vertices must be in the kernel set. Thus $\kappa_a \geqslant \lceil\frac{k}{2}\rceil+1$. Now $\{u, x_{m-(8i-6)}, x_{m-(8i-3)}\}$, $i=1,2,...,\lceil\frac{k}{4}\rceil$ is a kernel for $F(m,k)$ which forms an independent set. Thus $\kappa_a = \lceil\frac{k}{2}\rceil+1$. See Figure 2.

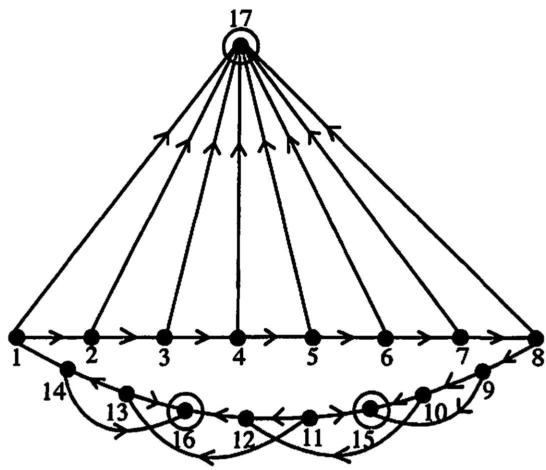

Figure 2: Encircled vertices form a kernel in $F(16,4)$

Case 2: $\lceil\frac{k}{2}\rceil$ is odd: Define a labeling $f: V(G) \to \{1,2,...,m,m+1\}$ by $f(u)=m+1$, $f(x_{m-(8i-6)}) = \{m, m-1, ..., m-\lceil\frac{k}{4}\rceil+1\}$ where $i=1,2,...,\lceil\frac{k}{4}\rceil$ and $f(x_{m-(8j-3)}) = \{m-\lceil\frac{k}{4}\rceil, ..., m-\lceil\frac{k}{2}\rceil+1\}$ where $j=1,2,...,\lceil\frac{k}{4}\rceil-1$ Let the remaining vertices be labeled as $1,2,...,m-\lceil\frac{k}{2}\rceil$ arbitarily by f. Orient the edges $(\overrightarrow{v,w}) \in E(G)$ if and only if $f(v) < f(w)$, rendering f a topological ordering. Then by theorem 1, G is acyclic. We claim that vertices labeled
$m+1, m, m-1, ..., m-\lceil\frac{k}{2}\rceil+1$ constitute an acyclic kernel set K of G. Since u be a vertex of degree $m-2k$, u covers atmost $m-2k$ vertices of G. All the remaining $2k$ vertices of G are of degree 3. Thus at least $\lceil\frac{k}{2}\rceil+1$ vertices must be in the kernel set. Thus $\kappa_a \geqslant \lceil\frac{k}{2}\rceil+1$. Now $\{u, x_{m-(8i-6)}, x_{m-(8j-3)}\}$, $i=1,2,...,\lceil\frac{k}{4}\rceil$ and $j=1,2,...,\lceil\frac{k}{4}\rceil-1$ is a kernel for $F(m,k)$ which forms an independent set. Thus $\kappa_a = \lceil\frac{k}{2}\rceil+1$.

Theorem 3 *The acyclic kernel problem for the circular fan $F(m,k)$ with $k \geqslant 2$ chords and $m \geqslant 8$ is polynomially solvable.*

5 Kernel in Double Headed Circular Fan

Definition 9 *Let $C : x_1x_2...x_mx_1$ be a cycle on m vertices. For $x_i \in V(C)$, the graph obtained by adding edges (x_i,u), $i = 1,2,...,m-3$ and (x_i,v), $i = m-2, m-1, m$ to C is called Double headed circular fan. It is denoted by $DF(m)$. The new edges are called spokes of $DF(m)$.*

Theorem 4 *Let G be the double headed circular fan $DF(m)$, $m \geqslant 7$. Then $\kappa_a = 2$.*

Proof. Define a labeling $f : V(G) \to \{1,2,...,m,m+1,m+2\}$ by $f(u) = m+2$, $f(v) = m+1$ and $f(x_m) = m$. Orient the edges $(\overrightarrow{v,w}) \in E(G)$ if and only if $f(v) < f(w)$ rendering f a topological ordering of $V(G)$ such that the vertices labeled $m+2$, $m+1$ have in-degree $m-3$ and 3 respectively. Then by theorem 1, G is acyclic.

We claim that vertices labeled $m+2$, $m+1$ constitute an acyclic kernel set K of G.

Since u is a vertex of degree $m-3$, u covers atmost $m-3$ vertices of G. All the remaining four vertices of G are of degree 3. Thus at least one of these vertices must be in the kernel set. Thus $\kappa_a \geqslant 2$. Now $\{u,v\}$ is a kernel set for $DF(m)$ which forms an independent set. Thus $\kappa_a = 2$. See Figure 3.

Theorem 5 *The acyclic kernel problem for the double headed circular fan $DF(m)$, $m \geqslant 7$ is polynomially solvable.*

6 Kernel in Double Headed Circular Fan with k Chords $DF(m,k)$

Definition 10 *A Double fan is defined as $P_n + \overline{K_2}$.*

It is easy to prove the following result.

Theorem 6 *Let G be a double fan. Then $\kappa_a = 2$.*

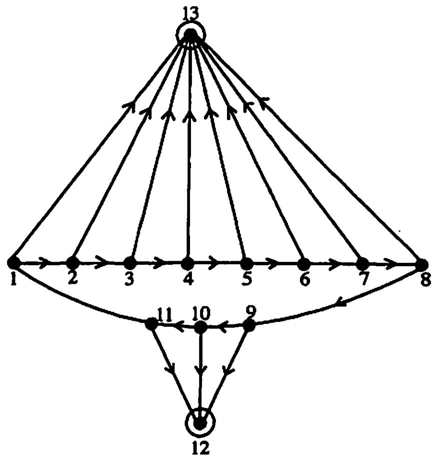

Figure 3: $DF(10)$ with kernel vertices encircled

Definition 11 *Let $C : x_1x_2...x_mx_1$ be a cycle on m vertices. For $x_i \in V(C)$, the graph obtained by adding*
(a) edges (x_i, u), $i = 1, 2, ..., m-(2k+3)$ and (x_i, v), $i = m-(k+3), m-(k+2), m-(k+1)$ to C and chords (x_m, x_{m-2}), $(x_{m-2k}, x_{m-(2k+2)})$, $(x_{m-(k-1)}, x_{m-(k+5)})$ and $(x_{m-i}, x_{m-(i+3)})$, where $i = 1, 3, ..., \lfloor \frac{2k-5}{2} \rfloor$ and $i = k+4, k+6, ..., 2k-2$ when k is even or by adding
(b) edges (x_i, u), $i = 1, 2, ..., m-(2k+3)$ and (x_i, v), $i = m-(k+4), m-(k+3), m-(k+2)$ to C and chords (x_m, x_{m-2}), $(x_{m-2k}, x_{m-(2k+2)})$, $(x_{m-k}, x_{m-(k+6)})$, $(x_{m-(k-2)}, x_{m-(k+1)})$ and $(x_{m-i}, x_{m-(i+3)})$, where $i = 1, 3, ..., \lfloor \frac{2k-5}{2} \rfloor - 1$ and $i = k+5, k+7, ..., 2k-2$ when k is odd
is called a Double headed circular fan with k chords where $k \geqslant 2$ and is denoted by $DF(m, k)$. The new edges are called spokes of $DF(m, k)$. See Figure 4.

Theorem 7 *Let G be $DF(m, k)$ with $k \geqslant 2$ chords and $m \geqslant 11$. Then $\kappa_a \geqslant \lceil \frac{k}{2} \rceil + 2$.*

Proof. Let G be $DF(m, k)$ with $k \geqslant 2$ chords and $m \geqslant 11$. Since u is a vertex of degree $m-(2k+3)$, u covers atmost $m-(2k+3)$ vertices of

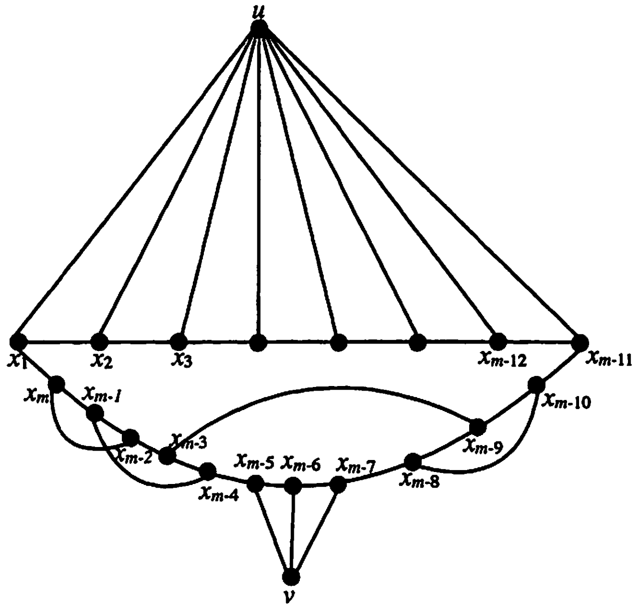

Figure 4: $DF(m,4)$

G. All the remaining $2k+3$ vertices of G are of degree 3. Thus at least $\lceil \frac{k}{2} \rceil + 2$ of these vertices must be in the kernel set. Hence $\kappa_a \geqslant \lceil \frac{k}{2} \rceil + 2$.

Theorem 8 *The acyclic kernel problem for the double headed circular fan with k chords $DF(m,k)$ is polynomially solvable.*

7 Conclusion

We have discussed the acyclic kernel number problem for oriented graphs. In this paper, we have proved that the acyclic kernel problem for the circular fan with k chords $F(m,k)$, double headed circular fan $DF(m)$, $m \geqslant 7$ and double headed circular fan with k chords $DF(m,k)$ are polynomially solvable. Further the acyclic kernel problem for circular ladders and Petersen graphs are under investigation.

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