Packing of Hexagonal Networks

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Abstract

In cellular radio communication systems, the concept of maximum packing is used for dynamic channel assignment. An H-packing of a graph G is a set of vertex disjoint subgraphs of G, each of which is isomorphic to a fixed graph H. The maximum H-packing problem is to find the maximum number of vertex disjoint copies of H in G called the packing number denoted by $\lambda(G, H)$. In this paper we determine the maximum H-packing number of hexagonal networks when H is isomorphic to P_6 as well as $K_{1,3}$.

Keywords: H-packing, packing number, matching, hexagonal networks

1 Introduction

The maximum packing policy for dynamic channel assignment in cellular radio communication systems specifies that a new call attempt is admitted whenever there is some way of rearranging channels so that every call can be carried. Otherwise the call is blocked and removed from the system [11]. In cellular radio communication systems, maximum packing policy is used for dynamic channel assignment.

An *H-packing* of a graph G is a set of vertex disjoint subgraphs of G, each of which is isomorphic to a fixed graph H. The maximum H-packing problem is to find the maximum number of vertex disjoint copies of H in G called the *packing number* denoted by $\lambda(G, H)$. For our convenience $\lambda(G, H)$ is sometimes represented as λ. An H-packing in G is called *perfect* if it covers all vertices of G. An *F-packing* is a natural generalization of H-packing concept. For a given family F of graphs, the problem is to identify a set of vertex-disjoint subgraphs of G, each isomorphic to a member of F. The F-packing problem is to find an F-packing in a graph G that covers the maximum number of vertices of G.

If H is the complete graph K_2, the maximum H-packing problem becomes the familiar maximum matching problem. H-Packing, is of practical interest in the areas of scheduling [1], wireless sensor tracking [2],

*This work is supported by DST Project No. SR/S4/MS: 595/09.

wiring-board design, code optimization [7] and many others. When H is a connected graph with at least three vertices, Kirkpatrick and Hell proved that the maximum H-packing problem is NP-complete [7]. Packing lines in a hypercube has been studied in [6]. Algorithms are available for dense packing of trees of different sizes [17] and packing almost stars [5] into the complete graph. The H-packing problem when H is a tree and, in particular, when H is a path of two edges has been studied by Kelmans et al. [9].

One of the most widely studied packing is claw-packing [5]. A claw is another name for the complete bipartite graph $K_{1,3}$. A claw-free graph is a graph in which no induced subgraph is a claw. The packing of induced stars in a graph has been studied in [10]. Las Vergnas proved that the $\{S_1, ..., S_k\}$-packing problem where $S_t \simeq K_{1,t}$ is polynomially solvable [16]. On the contrary, Hell and Kirkpatrick [8] proved that the packing problem when $F = \{S_i : i \in J\}$ is NP-complete whenever $J \subseteq N$ is not of the form $\{1, 2, ..., k\}$. In this paper we study the packing of hexagonal networks with S_3.

2 Hexagonal Network

In a direct interconnection network, nodes represent processors, while edges indicate connections between processors for direct message exchange. A survey of such networks is given in [14]. It is known that there exist three regular plane tessellations, composed of the same kind of regular polygons: triangular, square, and hexagonal. They are the basis for the design of direct interconnection networks with highly competitive overall performance.

Hexagonal networks are based on regular triangular tessellations, or the partition of a plane into equilateral triangles and are widely studied in [3]. Hexagonal network $HX(n)$ of dimension n has $3n^2 - 3n + 1$ vertices and $9n^2 - 15n + 6$ edges, where n is the number of vertices on one side of the hexagon [3]. See Figure 1. The diameter is $2n - 2$. There are six vertices of degree three which we call as *corner vertices*.

There is exactly one vertex v at distance $n - 1$ from each of the corner vertices. This vertex is called the centre of $HX(n)$ and is represented by O. The vertex set V is partitioned into sets inducing concentric cycles around O. Call vertex O as level 1, the first cycle around O as level 2 denoted by C_2^o and so on and the last cycle farthest from O as level n denoted by C_n^o. See Figure 1. The level i cycle has $6(i - 1)$ vertices, $i \geq 2$.

Hexagonal networks are studied in a variety of contexts. They are ap-

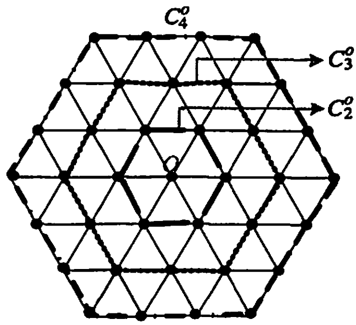

Figure 1: $HX(4)$

plied in chemistry to model benzenoid hydrocarbons [15], in image processing and computer graphics [12], and wireless and interconnection networks. An addressing scheme for processors, and corresponding routing and broadcasting algorithms for hexagonal interconnection network were proposed by Chen et al. [3]. The performance of hexagonal networks was further studied in [4, 13]. In the sequel let C_n and P_n denote a cycle and a path on n vertices respectively.

Definition 1 *The subgraph induced by C_i^o and C_{i-1}^o in $HX(n)$ is called a circular channel and is denoted by $CC(i)$ for $i = 3, 5, \ldots, n$ if n is odd and for $i = 2, 4, \ldots, n$ if n is even. $CC(i)$ contains $6(2i-3)$ vertices, $i \geq 3$ and $CC(2)$ contains 7 vertices. See Figure 2(a) and 2(b).*

The vertices in $CC(k)$ are labelled as shown in the Figure 2. x_1^k, x_2^k, ..., x_{12k-18}^k shown in Figure 2 are consecutive vertices in the circular channel $CC(k)$.

Theorem 1 *Let G be a graph and H be a subgraph of G. Then $\lambda(G, H) \leq \left\lfloor \frac{|V(G)|}{|V(H)|} \right\rfloor$.*

Proof. It is clear that λ number of vertex disjoint copies of H in G cover $\lambda(G, H) \times |V(H)|$ distinct vertices of G.

Therefore $\lambda(G, H) \times |V(H)| \leq |V(G)|$. □

Remark 1 *Since $|V(HX(n))|$ is a prime number, no subgraph will perfectly pack $HX(n)$. We note that $|V(HX(n))| - 1$ is always a multiple of 6. In the next section we pack $HX(n)$ with P_6.*

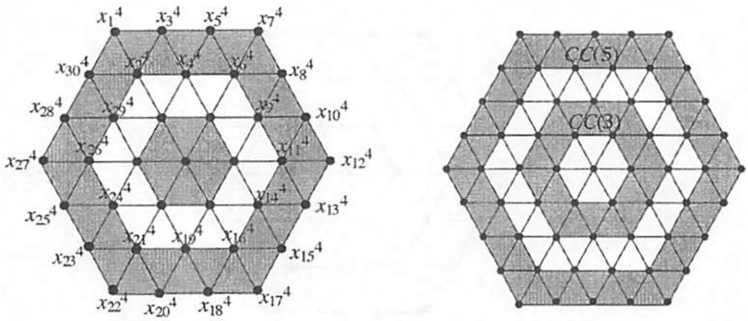

Figure 2: Circular channels in $HX(4)$ and $HX(5)$

3 Packing of $HX(n)$ with P_6

In this section we study the packing of $HX(n)$ with P_6.

Theorem 2 *There exists a perfect H-Packing of $HX(n)$ with* $\left\lfloor \frac{3n^2-3n+1}{6} \right\rfloor$ *copies of* $H \cong P_6$.

Proof. By Theorem 1, $\lambda(G,H) \leq n^2$. Now for $k = 2,\ldots,n$ $C_k^o \simeq C_{6k-6}$. Let $V(C_k^o) = \{1,2,...,6k-6\}$. Then $S_t = \{6t+1, 6t+2, 6t+3, 6t+4, 6t+5, 6t+6\}$ where $0 \leq t \leq k-2$ is a partition of C_k^o into paths of length 6. Therefore $\lambda(G,H) \geq 1+2+\ldots+(n-1) = \frac{n(n-1)}{2} = \left\lfloor \frac{3n^2-3n+1}{6} \right\rfloor$. Thus $\lambda(G,H) = \left\lfloor \frac{3n^2-3n+1}{6} \right\rfloor$. □

Remark 2 *If a graph G is packed by P_n then G is also packed by P_d for all divisors d of n.*

It follows that $HX(n)$ can also be packed by P_2 and P_3. Since packing $HX(n)$ with P_2 is nothing but perfect matching, we have the following theorem.

Theorem 3 *There exists a perfect matching in $HX(n)$.*

3.1 Packing $HX(n)$ with Claw

In this section we describe an efficient algorithm that perfectly packs $HX(n)$ with a claw.

Procedure PACKING $(HX(n), K_{1,3})$

Input: A hexagonal network G **of dimension** n **and** $H \simeq K_{1,3}$.

Algorithm: Let $k = n$.
Mark x_1^k saturated.
While $k \geq 3$ Do
While $j > 12k - 19$ Do

(i) Having marked x_i^k move along consecutive vertices in $CC(k)$ till we arrive at x_j^k such that $(|N[x_i^k] \cap N[x_j^k]| \leq 1$ and $d(x_j^k) = 4)$ or $(|N[x_i^k] \cap N[x_j^k]| \leq 2$ and $d(x_j^k) = 5)$ or $(|N[x_i^k] \cap N[x_j^k]| = 2$, $d(x_j^k) = 4$, $d(x_i^k) = 5$ and $|N[x_i^k] \cap N[x_h^k]| = 1$ where x_h^k is already marked). Mark x_j^k saturated.

Repeat

(ii) Mark $w = x_{12k-19}^k$ saturated if $x_{12k-19}^k \notin N[v]$, for any saturated vertex v.

(iii) $k \longleftarrow k - 2$.

(iv) Mark x_1^k or x_3^k saturated according as w is unsaturated or saturated respectively.

Repeat

(v) Mark O saturated if $n \equiv 0 \bmod 4$ and mark x_1^2 saturated if $n \equiv 2 \bmod 4$.

End PACKING

Output: An H-packing of $HX(n)$ with $\left\lfloor \frac{3n^2-3n+1}{4} \right\rfloor$ copies of $K_{1,3}$. See Figure 3.

Proof of Correctness: The subgraph induced by $N[v]$ when v is a saturated vertex contains a subgraph isomorphic to $K_{1,3}$. For $u \neq v$, either $N[u] \cap N[v] = \Phi$ or there exist $H_1 \simeq K_{1,3}$ in $N[u]$ and $H_2 \simeq K_{1,3}$ in $N[v]$ such that $H_1 \cap H_2 = \Phi$. For $k > 1$, $CC(k)$ contains $\lfloor \frac{12k-18}{4} \rfloor$ number of saturated vertices. Exactly one copy of $K_{1,3}$ shares vertices from successive channels $CC(k)$ and $CC(k-2)$. Therefore $\lambda(G, H) = \sum \lfloor \frac{12k-18}{4} \rfloor + \lfloor \frac{n}{4} \rfloor$. □

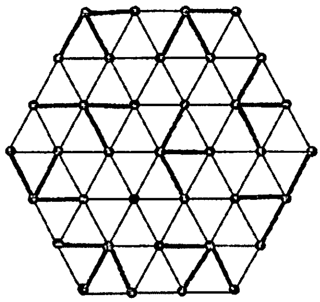

Figure 3: Illustrating the procedure packing $K_{1,3}$ in $HX(4)$

4 Conclusion

In this paper we determine the H-packing of hexagonal networks with H isomorphic to either P_6 or $K_{1,3}$. The H-packing problem when H is isomorphic to $K_{1,4}$, $K_{1,5}$ or $K_{1,6}$ and the F-packing problem when $F = \{S_i : i \in J\}$ whenever $J \subseteq N$ is not of the form $\{1, 2, \ldots, k\}$ are under investigation.

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