On the Crossing Number of Honeycomb Related Networks

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Abstract

The crossing number of a graph G is the minimum number of crossings of its edges among the drawings of G in the plane and is denoted by $cr(G)$. In this paper we obtain bounds for the crossing number for two different honeycomb tori namely, the honeycomb rectangular torus and the honeycomb rhombic torus which are obtained by adding wraparound edges to honeycomb meshes.

Keywords: crossing number, honeycomb rectangular torus, honeycomb rhombic torus.

1 Introduction

Crossing number minimization is one of the fundamental optimization problems in the sense that it is related to various other widely used notions. Besides its mathematical interest, there are numerous applications, most notably those in VLSI design [2, 8, 9] and in combinatorial geometry [18]. The study of crossing numbers of graphs also finds applications in areas of network design and circuit layout. Minimizing the number of wire crossings in a circuit greatly reduces the chance of cross-talk in long crossing wires carrying the same signal and also allows for faster operation and less power dissipation. We refer to [15, 19] for more details about such applications.

A drawing D of a graph G is a representation of G in the Euclidean plane R^2 where vertices are represented as distinct points and edges by simple polygonal arcs joining points that correspond to their end vertices. A drawing D is *good* or *clean* if it has the following properties.

1. No edge crosses itself.

2. No pair of adjacent edges cross.

3. Two edges cross at most once.

4. No more than two edges cross at one point.

The number of crossings of D is denoted by $cr(D)$ and is called the *crossing number* of the drawing D. The crossing number $cr(G)$ of a graph G is the minimum $cr(D)$ taken over all good or clean drawings D of G. If a graph G admits a drawing D with $cr(D) = 0$ then G is said to be planar; otherwise non-planar. It is well known that K_5, the complete graph on 5 vertices and $K_{3,3}$ the complete bipartite graph with 3 vertices in its classes are non-planar. According to Kuratowski's famous theorem, a graph is planar if and only if contains no subdivision of K_5 or $K_{3,3}$.

The study of crossing numbers began during the Second World War with Paul Turán. For an arbitrary graph computing $cr(G)$ is NP-hard [6]. Hence from a computational standpoint, it is infeasible to obtain exact solutions for graphs, in greneral, but more practical to explore bounds for the parameter values [4]. Richter and Thomassen [14] discussed the relation between crossing numbers of the complete and the complete bipartite graphs. The bound for $cr(K_n)$ and $cr(K_{m,n})$ are obtained by Guy [7]. In particular, Pan et al. [12] have shown that $cr(K_{11}) = 100$ and $cr(K_{12}) = 153$. Nahas [11] has obtained an improved lower bound for $K_{m,n}$. In [5, 13] the crossing number of some generalized Petersen graphs $P(2n+1, 2)$ and $P(3k+h, 3)$ has been discussed. For hypercubes and cube connected cycles the crossing number problem is investigated by Sýkora et al. [17]. Cimikowski [4] has obtained the bound for the crossing number of mesh of trees. We have obtained the bounds for the crossing number for two different representations of the standard butterfly network [1].

Honeycomb tori have been recognized as an attractive alternative to existing torus interconnection networks in parallel and distributed applications. In this paper, we have obtained upper bounds for the crossing number for two different honeycomb tori namely, the honeycomb rectangular torus and the honeycomb rhombic torus.

2 Honeycomb Related Networks

Honeycomb meshes can be built from hexagons in various ways. The simplest way to define them is to consider the portion of the hexagonal tessellation which is inside a given convex polygon. Stojmenovic [16] considers three types of meshes which differ by their boundary. Honeycomb hexagon mesh (HHM) is inside a regular hexagon (Figure 1(a)), honeycomb rectangular mesh ($HReM$) is inside a rectangle (Figure 1(b)) and honeycomb rhombic mesh ($HRoM$) is inside a rhombus (Figure 1(c)). Stojmenovic [16] introduced three different honeycomb tori by adding wraparound edges on honeycomb meshes namely, the honeycomb rectangular torus, the honeycomb rhombic torus, and the honeycomb hexagonal torus.

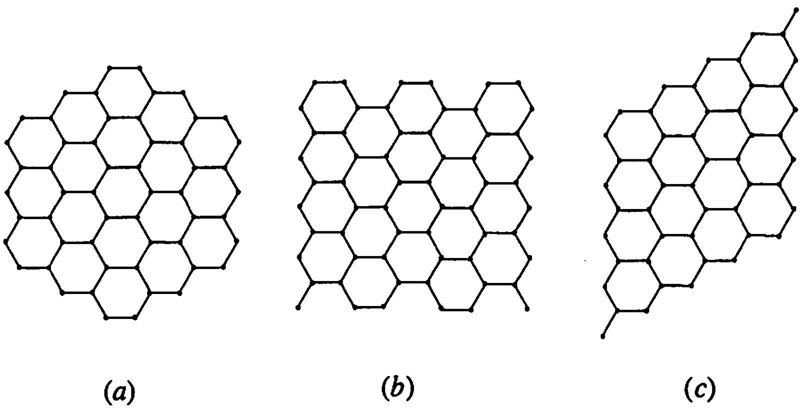

Figure 1: (a). Honeycomb hexagonal mesh (b). Honeycomb rectangular mesh (c). Honeycomb rhombic mesh

2.1 Honeycomb Rectangular Torus

The honeycomb rectangular mesh [20] $HReM(m,n)$ is the graph with vertex set

$$V(HReM(m,n)) = \{(i,j) : 0 \leq i < m, 0 \leq j < n\}$$

such that (i,j) and (k,l) are adjacent if they satisfy one of the following conditions:

1. $i = k$ and $j = l \pm 1$
2. $j = l$ and $k = i - 1$ if $i + j$ is even

where m and n are positive even integers. Figure 2 depicts a honeycomb rectangular mesh $HReM(4,4)$.

Honeycomb rectangular torus is a honeycomb rectangular mesh with wraparound edges. The honeycomb rectangular torus $HReT(m,n)$ is the graph with

$$V(HReT(m,n)) = \{(i,j) : 0 \leq i < m, 0 \leq j < n\}$$

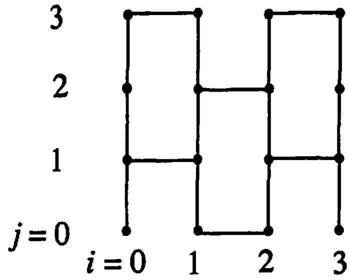

Figure 2: Honeycomb Rectangular Mesh $HReM(4,4)$

such that (i,j) and (k,l) are adjacent if they satisfy one of the following conditions:

1. $i = k$ and $j = l \pm 1 (\bmod n)$.
2. $j = l$ and $k = i - 1 (\bmod m)$ if $i + j$ is even.

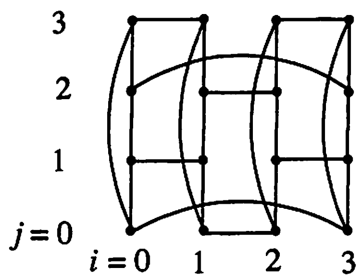

Figure 3: Honeycomb Rectangular Torus $HReT(4,4)$

Clearly, $HReT(m,n)$ has m vertical wraparound edges and $\frac{n}{2}$ horizontal wraparound edges. We call the remaining edges as straight edges. See Figure 3. It follows from Figure 4 that a subdivision of $K_{3,3}$ is contained in $HReT(4,4)$ and more so in any $HReT(m,n)$. By Kuratowski's theorem this means that the Honeycomb rectangular torus is non-planar. We therefore consider different drawings of $HReT(m,n)$ and determine the crossings in each case.

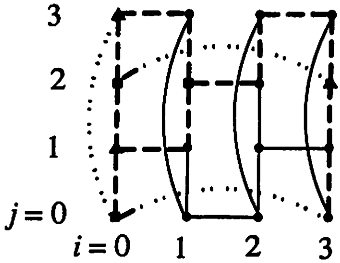

Figure 4: A subdivision of $K_{3,3}$ in $HReT(4,4)$

Lemma 1 *If no edge is drawn as exterior arc, then the number of crossings is* $\frac{3mn}{2} - m - n$.

Proof. $HReT(m,n)$ has m vertical wraparound edges and $\frac{n}{2}$ horizontal wraparound edges. Each of m vertical wraparound edges crosses $\frac{n}{2} - 1$ straight edges and contributes $\frac{n}{2} - 1$ crossings. Each of $\frac{n}{2}$ horizontal wraparound edges crosses $m-2$ straight edges and contributes $m-2$ crossings. Apart from this, the horizontal and vertical wraparound edges cross each other which will give rise to $\frac{1}{2}(mn)$ crossings. Hence the number of crossings is $\frac{3mn}{2} - m - n$. See Figure 5(a).

Lemma 2 *If the horizontal wraparound edges are drawn as exterior arcs, then the number of crossings is* $\frac{1}{2}(mn - 2m)$.

Proof. If the horizontal wraparound edges in $HReT(m,n)$ are drawn as exterior arcs, only vertical wraparound edges contribute for crossing. Each of these edges contributes $\frac{n}{2} - 1$ crossings. Hence the total number of crossings in this case is equal to $m(\frac{n}{2} - 1) = \frac{1}{2}(mn - 2m)$. See Figure 5(b).

Lemma 3 *If the vertical wraparound edges are drawn as exterior arcs, then the number of crossings is* $\frac{1}{2}(mn - 2n)$.

Proof. In this case, only horizontal wraparound edges contribute for crossing. Each of these edges contributes $m-2$ crossings. Hence the total number of crossings in this case is equal to $\frac{n}{2}(m-2) = \frac{1}{2}(mn - 2n)$. See Figure 5(c).

Lemma 4 *If all the wraparound edges are drawn as exterior arcs, then the number of crossings is* $\frac{1}{2}(mn)$.

Proof. Here the horizontal and vertical wraparound edges cross each other so that there are only $\frac{n}{2}m$ crossings. Hence the number of crossings is $\frac{1}{2}(mn)$. See Figure 5(d).

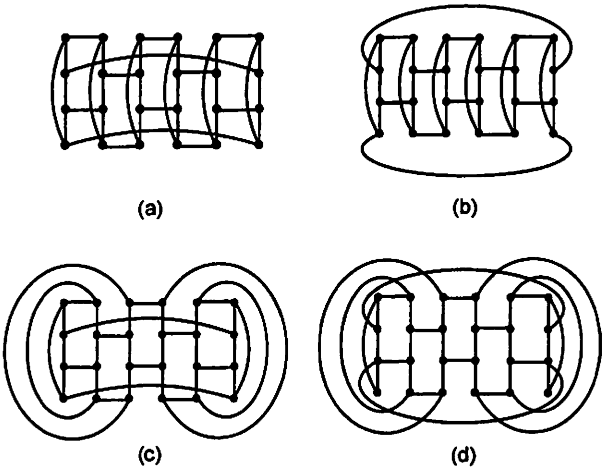

Figure 5: Different Drawings of $HReT(6,4)$

In view of lemma 2 and lemma 3, we have the following result.

Theorem 5 *Let $G = HReT(m,n)$ be a honeycomb rectangular torus. Then $cr(G) \le \frac{1}{2}(mn - 2x)$ where $x = max\{m,n\}$.*

2.2 Honeycomb Rhombic Torus

Assume that m and n are positive integers, where n is even. The honeycomb rhombic torus $HRoT(m,n)$ is the graph with

$$V(HRoT(m,n)) = \{(i,j) : 0 \le i < m, 0 \le j - i < n\}$$

such that (i,j) and (k,l) are adjacent if they satisfy one of the following conditions:

1. $i = k$ and $j = l \pm 1 (\text{mod } n)$.
2. $j = l$ and $k = i - 1$ if $i + j$ is even.
3. $i = 0$, $k = m - 1$, and $l = j + m$ if j is even.

$HRoT(m,n)$ has m vertical wraparound edges and $\frac{n}{2}$ diagonal wrap-around edges. Here also we consider different drawings of $HRoT(m,n)$ and determine the crossings in each case. See Figure 6.

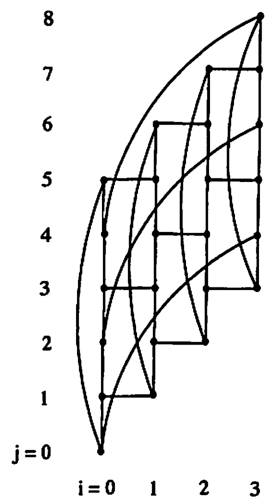

Figure 6: $HRoT(6,4)$

Lemma 6 *If no edge is drawn as exterior arc, then the number of crossings is* $\frac{3mn}{2} - m - \frac{n}{2}$.

Lemma 7 *If the vertical wraparound edges are drawn as exterior arcs, then the number of crossings is* $\frac{1}{2}(mn - n)$.

Lemma 8 *If the diagonal wraparound edges are drawn as exterior arcs, then the number of crossings is* $\frac{1}{2}(mn - 2m)$.

Lemma 9 *If all the wraparound edges are drawn as exterior arcs, then the number of crossings is* $\frac{1}{2}(mn)$.

The proofs of the above lemmas are similar to those of honeycomb rectangular torus. Thus we have the following result.

Theorem 10 *Let* $G = HRoT(m,n)$ *be a honeycomb rhombic torus. Then* $cr(G) \leq \frac{1}{2}(mn - x)$ *where* $x = max\{2m, n\}$.

3 Conclusion

We have obtained upper bounds for the crossing number of two different honeycomb tori namely, the honeycomb rectangular torus and the honeycomb rhombic torus. The crossing number problem for honeycomb hexagonal torus and generalized honeycomb torus are under investigation.

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