Embedding in Fat Trees

Indra Rajasingh[1,2], Bharati Rajan[2], R. Sundara Rajan[2], Paul Manuel[3]
[2]Department of Mathematics, Loyola College, Chennai 600 034, India
[3]Department of Information Science, Kuwait University, Safat, Kuwait
hellosundar@rediffmail.com

Abstract

We show that the butterfly network and benes network can be embedded into generalized fat trees with minimum dilation.

Keywords: Butterfly and Benes networks, generalized binary fat trees, embedding

1 Introduction

Graph embedding is an important technique used in the study of computational capabilities of processor interconnection networks and task distribution. Embeddings of graphs from one class of graphs into another class of graphs have important applications in computer science. For example, any finite graph can be considered as a model of a parallel computer, where vertices correspond to processors and edges represent communication lines between them [10].

The concept of embedding is widely studied in the area of fixed interconnection parallel architectures. A parallel architecture is embedded into another architecture to simulate one on another. An important feature of an interconnection network is its ability to efficiently simulate programs written for other architectures [10, 13].

Embeddings of graphs with a regular structure, like rings, grids, complete trees, binomial trees, pyramids, X-trees, meshes of trees and so on, have been investigated by numerous researchers [2, 12]. In general, the communication structure of a parallel algorithm can be very irregular. Embeddings of such irregular graphs, like binary trees, caterpillars, graphs with bounded treewidth, have also been studied [3, 15]. However, there has been no work reported so far on embeddings of butterfly and benes networks into generalized fat trees in the literature.

---

[1]This work is supported by DST Project No. SR/S4/MS: 494/07.

Corresponding author : Indra Rajasingh, Department of Mathematics, Loyola College, Chennai 600 034, India. E-mail : indrarajasingh@yahoo.com

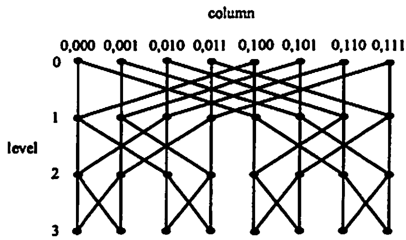

Figure 1: 3-dimensional Butterfly Network

Definition 1 *Let G and H be finite graphs with n vertices. $V(G)$ and $V(H)$ denote the vertex sets of G and H respectively. $E(G)$ and $E(H)$ denote the edge sets of G and H respectively. An embedding f of G into H is defined [2] as follows:*
1. f is a bijective map from $V(G) \to V(H)$
2. f is a one-to-one map from $E(G)$ to $\{P_f(f(u), f(v)) : P_f(f(u), f(v))$ is a path in H between $f(u)$ and $f(v)\}$.

The dilation of an embedding f of G into H is given by

$$dil(f) = \max\{|P_f(f(u), f(v))| : (u, v) \in E(G)\}$$

where $|P_f(f(u), f(v))|$ denotes the length of the path P_f. Then, the dilation of G into H is defined as

$$dil(G, H) = \min dil(f)$$

where the minimum is taken over all embeddings f of G into H.

Embedding G into H with minimum dilation is important for network design and for the simulation of one computer architecture by another [3].

Definition 2 *[5] The m-dimensional butterfly BF_m has $n = 2^m\ (m+1)$ nodes arranged in $m+1$ levels of 2^m nodes each. Each node has a distinct label $\langle w, i\rangle$ where i is the level of the node $(0 \le i \le m)$ and w is a m-bit binary number that denotes the column of the node. All nodes of the form $\langle w, i\rangle$, $0 \le i \le m$, are said to belong to column w. Similarly, the i^{th} level L_i consists of all of the nodes $\langle w, i\rangle$, where w ranges over all m-bit binary numbers. Two nodes $\langle w, i\rangle$ and $\langle w', i'\rangle$ are linked by an edge if $i' = i+1$ and*

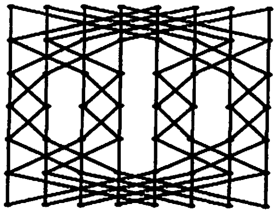

Figure 2: 3-dimensional Benes Network

either w and w' are identical or w and w' differ only in the bit in position i'. (The bit positions are numbered 1 through m, the most significant bit being numbered 1). The edges in the network are undirected. The nodes on level 0 are called the input nodes or just inputs of the network, and the nodes on level m are called the output nodes or just outputs. See Figure 1.

Definition 3 *[11] An m-dimensional Benes network B_m has $2m+1$ levels, each level with 2^m nodes. The level 0 to level m nodes in the network form an m dimensional butterfly. The middle level of the Benes network is shared by these butterflies [10]. Figure 2 shows a B_3 network.*

Leiserson [9] proposed fat trees as a hardware-efficient, general-purpose interconnection network. Several architectures including the Connection Machine CM-5 of Thinking Machines, the memory hierarchy of the KSR-1 parallel machine of Kendall Square Research [6], and Meiko supercomputer CS-2 [8] are based on the fat trees. A different fat tree topology called "pruned butterfly" is proposed in [1], and other variants are informally described in [7], where the increase in channel bandwidth is modified compared to the original fat trees [9].

The generalized fat tree $GFT(h, m, w)$ [14] of height h consists of m^h processors in the leaf-level and routers or switching-nodes in the non-leaf levels. Each non-root has w parent nodes and each non-leaf has m children. Informally, $GFT(h+1, m, w)$ is recursively generated from m distinct copies of $GFT(h, m, w)$, denoted as $GFT^j(h, m, w) = (V_h^j, E_h^j)$, $0 \leq j \leq m-1$, and w^{h+1} additional nodes such that each top-level node $(h, k + j \cdot w^h)$ of each $GFT^j(h, m, w)$ for $0 \leq k \leq w^h - 1$, is adjacent to w consecutive new top-level nodes (*i.e.* level $h+1$ nodes), given by $(h+1, k \cdot w), \ldots, (h+1, (k+1) \cdot w - 1)$. The graph $GFT^j(h, m, w)$ is also called a sub-fat tree of $GFT(h+1, m, w)$.

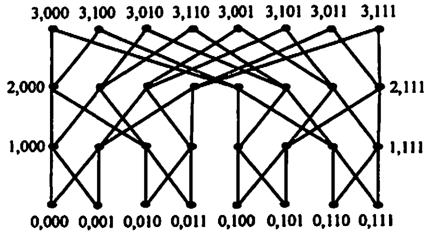

Figure 3: 3-dimensional Generalized Binary Fat Tree

In this paper we call $GFT(h,2,2)$ a binary fat tree and denote it by $BFT(h)$. $GFT(h,2,2)$ is precisely defined as follows.

Definition 4 *The m-dimensional generalised binary fat tree $BFT(m)$ has $n = 2^m\ (m+1)$ nodes arranged in $m+1$ levels of 2^m nodes each. Each node has a distinct label $\langle z, j\rangle$ where j is the level of the node $(0 \leq j \leq m)$ and $z = (a_m...a_j...a_2a_1)$ is a m bit binary number. Two nodes $\langle z, j\rangle$ and $\langle z', j'\rangle$ are adjacent if $j' = j+1$ and either w and w' are identical or $z' = (a_m...\overline{a_j}...a_2a_1)$. The edges in the network are undirected. The nodes on level 0 are called the input nodes or just inputs of the network, and the nodes on level m are called the output nodes or just outputs. See Figure 3.*

In this paper we will find the dilation of embedding BF_m and B_m into $GFT(h,m,w)$.

2 Main Results

Theorem 1 *Any graph G can be embedded into its optimal generalized fat tree $GFT(h,m,w)$ with dilation $2h$.*

Proof. Choose any edge $e = (a,b) \in G$. Without loss of generality, let a be mapped to $(0; a_ha_{h-1}...a_1)$ and b to $(0; b_hb_{h-1}...b_1)$. It is sufficient to give a path between the node $(0; a_ha_{h-1}...a_1)$ and the top-level node $t = (h; 0, 0, ..., 0)$ and from t to the node $(0; b_hb_{h-1}...b_1)$.

A possible path form $(0; a_ha_{h-1}...a_1)$ to $(h; 0, 0, ..., 0)$ is the following path:

$$P = \{(0; a_h...a_1), (1; a_h...a_20), (2; a_h....a_300), \cdots, (h-1; a_h0...0), (h; 0...0)\}$$

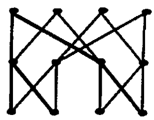

Figure 4: BF_1 can be embedded into $BFT(2)$

Again, the w number of node-disjoint paths P_i, $0 \leq i \leq w-1$, between the node a and b can be given as:

$$\begin{aligned} P_i &= \{(0; a_h...a_1), (1; a_h...a_2 i), (2; a_h....a_3 0i), \cdots, (h-1; a_h 0...0i), \\ &\quad (h; 0...0i), (h-1; b_h 0...0i), (h-2; b_h b_{h-1} 0...0i), \cdots, (1; b_h...b_2 i), \\ &\quad (0; b_h...b_2 b_1)\}. \end{aligned}$$

Thus $|P_i| = 2h$ for all $i, 0 \leq i \leq w-1$. Hence the dilation is $2h$. Since every path between $(0; a_h a_{h-1}...a_1)$ to $(h; 0, 0, ..., 0)$ must passes through a top level node, there is no path of length less than $2h$ between these nodes. Therefore the dilation is $2h$. □

Remark 1 *The path P_i, $i = 0, 1, ..., w-1$ are node disjoint.* □

Theorem 2 *The m-dimensional Butterfly BF_m can be embedded into*

$BFT(r)$, where

$$r = \begin{cases} \text{height of } (BFT(r-1)) + 2, \text{ if } m = 2^l \\ \text{height of } (BFT(r-1)) + 1, \text{ otherwise} \end{cases}$$

for some l, with load one and dilation $2r$, where as BF_1 can be embedded into $BFT(2, 2, 2)$ with dilation 4.

Proof. We prove this result by induction on m. Figure 4 illustrates the embedding of BF_1 into $BFT(2)$. Assume the result to be true when $k = m$.

Consider $k = m+1$. The nodes in the Butterfly are connected in the fat tree using the path which routes over the least common ancestor in the fat tree. If there are two unused least common ancestors, one chooses the left one. Thus, the load is equal to one and dilation is equal to the diameter $2r+4$ or $2r+2$ according as $m = 2^l$ or otherwise. □

Theorem 3 *The m-dimensional Benes B_m can be embedded into $BFT(r)$, where*

$$r = \begin{cases} \textit{height of } (BFT(r-1)) + 2, \textit{ if } m = 2^l \\ \textit{height of } (BFT(r-1)) + 1, \textit{ if otherwise} \end{cases}$$

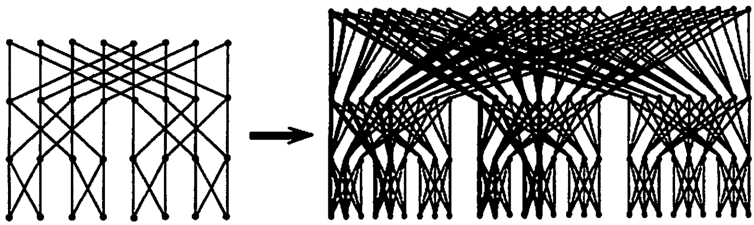

Figure 5: BF_3 can be embedded into $GFT(3,3,3)$

for some l, with load one and dilation $2r$, where as B_1 can be embedded into $BFT(3)$ with dilation 6. □

Note: In [14] the processors are considered at the leaf-level of $GFT(h,m,w)$. If we consider all the nodes as processors, then we get the following results.

Theorem 4 *BF_m is isomorphic to $BFT(m)$.*

Proof. For each vertex $\langle w,i\rangle$ in BF_m, we define a function g from $V(BF_m)$ to $V(BFT(m))$ as follows:

$$g(\langle w,i\rangle) = \langle w,i\rangle.$$

The function g is obviously bijective. Let $u = \langle w_1,i_1\rangle$ and $v = \langle w_2,i_2\rangle$ be two distinct vertices in BF_m. It follows that $g(u)$ and $g(v)$ are two distinct vertices in $BFT(m)$ given as follows:

$$g(u) = \langle w_1,i_1\rangle, g(v) = \langle w_2,i_2\rangle.$$

Vertices u and v are adjacent in $BF_m \iff i_2 = i_1+1$ and either w_1 and w_2 are identical or w_1 and w_2 differ only in the bit in position $i_2 \iff g(u)$ and $g(v)$ are adjacent in $BFT(m)$. Hence the graphs BF_m and $BFT(m)$ are isomorphic. □

The following results are easy consequence of the definition of generalized fat tree.

Theorem 5 *$GFT(h,m,w)$ is a subgraph of $GFT(h+1,m,w)$.* □

Theorem 6 *If $m_1 \geq m$, $w_1 \geq w$, then $GFT(h,m,w)$ is a subgraph of $GFT(h,m_1,w_1)$. □$,m_1,w_1)$. □$,m_1,w_1)$.* □

Corollary 1 *If $h \geq k$ and $m, w > 1$, then BF_k can be embedded into $GFT(h, m, w)$. See Figure 5.*

Conjucture: B_m can be embedded into $BFT(m+1)$ with dilation at least $m+1$. □

3 Conclusion

In this paper we have proved that the dilation of embedding BF_m and B_m into $GFT(h, m, w)$ is $2h$ and the dilation of embedding BF_m into $BFT(h)$ is h. It would be a good line of research to prove the conjuctures cited in this paper.

References

[1] P. Bay and G. Bilardi, Deterministic on-line routing on area universal networks. In Proceedings of the Annual Symp. on Foundations of Computer Science, (1990) 297 - 306.

[2] S.L. Bezrukov, J.D. Chavez, L. H. Harper, M. Röttger and U.P. Schroeder, Embedding of hypercubes into grids, MFCS (1998), 693 - 701.

[3] S.L. Bezrukov, B. Monien, W. Unger and G. Wechsung, Embedding ladders and caterpillars into the hypercube, Discrete Applied Mathematics, 83 (1998) 21 - 29.

[4] A. Bouabdallah, M.-C. Heydemann, J. Opatrny and D. Sotteau, Embedding complete binary trees into star and pancake graphs, Submitted to Mathematical Systems Theory, 1993.

[5] C. Bornstein, A. Litman, B. Maggs, R. Sitaraman and T. Yatzkar, On the bisection width and expansion of butterfly networks, Proceedings of the 12th International Parallel Processing Symposium, (1998) 144 - 150.

[6] S. Frank, J. Rothnie, and H. Burkhardt, The KSRl: Bridging the gap between shared memory and mpps, In Proceedings Compcon'93, San Francisco, CA, (1993) 285 - 294.

[7] R.I. Greenberg and C.E. Leiserson, Randomized routing on fat trees, In Silvio Micali, editor, Advances in Computing Research, Book 5: Randomness and Computation, JAI Press, Greenwich, (1989) 345 - 374.

[8] K.E. Schauser and C.J. Scheiman, Experiments with active messages on the Meiko CS-2. To appear in the Proceedings of the 9^{th} International Parallel Processing Symposium, Santa Barbara, (1995) 140 - 149.

[9] C.E. Leiserson, Fat-trees: Universal networks for hardware efficient supercomputing, IEEE Transactions on Computers, C-34, (1985) 892 - 901.

[10] T.F. Leighton, Introduction to Parallel Algorithms and Architecture: Arrays, Trees, Hypercubes, Morgan Kaufmann Publishers, San Mateo, Calif., 1992.

[11] P.D. Manuel , M.I. Abd-El-Barr, I. Rajasingh, B. Rajan, An Efficient representation of Benes networks and its applications, in: Proc. of the Sixteenth Australasian Workshop on Combinatorial Algorithms, (2005) 217 - 229.

[12] P. Manuel, I. Rajasingh, B. Rajan and H. Mercy, Exact Wirelength of Hypercube on a Grid, Discrete Applied Mathematics, 157, no.7, (2009) 1486 - 1495.

[13] B. Monien and H. Sudborough, Embedding one interconnection network in another, Computing Suppl., 7, (1990) 257-282.

[14] S.R. Öhring, M. Ibel, S.K. Das and M.J. Kumar, On generalized fat trees, IPPS, Proceedings of the 9^{th} International Symposium on Parallel Processing, IEEE Computer Society Washington, DC, USA, (1995) 37.

[15] I. Rajasingh, B. Rajan and S. Rajan, On embedding of m-sequencial k-ary trees into hypercubes, Applied Mathematics, 1, no.6, (2010) 499 - 503.

[16] J.M. Xu, Topological Structure and Analysis of Interconnection Networks, Kluwer Academic Publishers, 2001.