Betweenness Centrality of Honeycomb Networks

Indra Rajasingh, Bharati Rajan, Florence Isido. D
Department of Mathematics, Loyola College, Chennai 600 034, India.
Email: flornepo66@yahoo.co.in

Abstract

Betweenness is a centrality measure based on shortest paths, widely used in complex network analysis. *Betweenness centrality* of a vertex is defined as the fraction of shortest paths that pass through that vertex.over that between all pairs of vertices It measures the control a vertex has over communication in the network, and can be used to identify key vertices in the network. High centrality indices indicate that a vertex can reach other vertices on relatively short paths, or that a vertex lies on a considerable fraction of shortest paths connecting pairs of other vertices. In this paper we find betweenness centrality of honeycomb mesh which has important applications in mobile networks.

Keywords: interconnection network, honeycomb mesh, betweenness centrality.

1 Introduction

Real-world networks have been a field of study and research for a long time. They are represented by a graph $G = (V, E)$ of vertices V and edges E. A common example is the internet router topology where routers are vertices and links between routers are edges. One would like to know which routers or which links are *important*, e.g. how *severe* is the breakdown of a specified router or link. So *centrality measures* are required to label each vertex or edge with a number indicating its importance. But there is neither a mathematical definition for *important* nor for *severe*. So since 1950's many centrality indices have evolved, each with specific applications. Some applications include the facility location problem, highway-node routing, web page ranking or prediction of polls. A *centrality index* is a structural index for vertices or edges, which is based on shortest paths. Some centrality indices are closeness centrality, stress centrality, graph centrality, reach centrality and betweenness centrality. Of these measures, betweenness has been extensively used in recent years for the analysis of social interaction networks, as well as other large-scale complex networks [1]. It is computationally expensive to determine betweenness exactly; currently the fastest known algorithm by Brandes requires $O(nm)$ time for unweighted graphs and $O(nm + n^2 logn)$ time for weighted graphs, where n is the number of vertices and m is the number of edges in the network [2]. In this paper we calculate the exact betweenness centrality of vertices and a set of vertices which induces a hexagon in a honeycomb network.

2 An Overview of the Paper

First introduced in its modern form by Freeman [6], betweenness centrality is essentially a measure of how many geodesic paths pass through a given vertex. In otherwords, in a social network for example, the betweenness centrality measures the extent to which an actor "lies between" other actors in the network, with respect to the network path structure. As such, it is a measure of the control that actor has over the flow of information in the network. For large graphs, for instance a street graph of western Europe with approximately 18 million vertices and 22 million edges, exact calculation is almost unfeasible with only a small amount of time. Bader and Madduri [1] introduced parallel algorithms to calculate exact betweenness centrality.

There are two ways in which one might naturally extend vertex betweenness centrality to sets of vertices. The first is to define the betweenness of a set in terms of geodesic paths that pass through atleast one of the vertices in the set, and the second, in terms of geodesic paths that pass through all vertices in the set. The former notion was introduced by Everett and Borgatti [5] called *group betweenness centrality.* The latter was introduced by Kolaczyk et.al [4] called *co-betweenness centrality.* In this paper we study the betweenness centrality measure of a vertex in a honeycomb mesh and extend it to group betweenness centrality of a hexagon in the network.

3 Honeycomb Mesh-An Interconnection Network

In an interconnection network where processors are interconnected according to a certain topology, communications among processors are accomplished by sending messages along interconnection links.

Honeycomb mesh is an interconnection network used in wireless networks and its dual is used for cellular phone station placement. It is also used for the representation of benzenoid hydrocarbons in chemistry, computer graphics and image processing. Moreover, the number of papers on honeycomb appearing in the literature now shows that it is an attractive network [8].

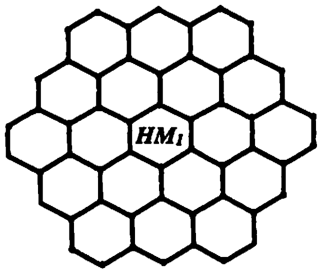

Figure 1: Honeycomb mesh HM_3 of dimension 3

Honeycomb meshes can be built from hexagons in various ways. The simplest way to define them is to consider the portion of the hexagonal tessellation which is inside a given convex polygon. Honeycomb (hexagonal) meshes can be built as follows: one hexagon is a honeycomb mesh of dimension one, denoted by HM_1. The honeycomb mesh HM_2 of dimension two is obtained by adding six hexagons to the boundary edges of HM_1. Inductively, honeycomb mesh HM_n of dimension n is obtained from HM_{n-1} by adding a layer of hexagons around the boundary of HM_{n-1}. For instance, Figure 1 is a honeycomb mesh of dimension three. Alternately, the dimension n of HM_n is determined as the number of hexagons between the center and boundary of HM_n (inclusive).

4 Honeycomb Parellelogram

We shall introduce a new convenient coordinate system for the honeycomb mesh. Let O be the centre of HM_1. Let us label the midpoints of edges of HM_1 in the clockwise direction as u_1, v_1, w_1, u_2, v_2 and w_2 as shown in Figure 2. Draw lines passing through u_1 and u_2, w_1 and w_2 and v_1 and v_2 and call them as the X-axis, Y-axis and Z-axis respectively. It is to be noted that the three axes X, Y, Z are at mutual angle of 120 degrees between any two of them and their point of intersection is O.

Delete all the edges of HM_n perpendicular to X-axis. This leaves n paths on either side of the X- axis, call them X-lines and label them

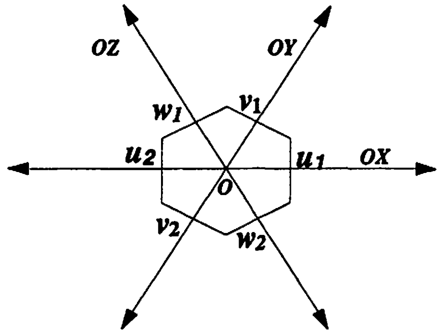

Figure 2: Honeycombmesh HM_1 of dimension 1

beginning from the path closest to X-axis sucessively to the paths farthest to X-axis as $X = 1, 3, ..., 2n - 1$ on one side of X-axis and as $X = -1, -3, ..., -(2n - 1)$ on the other side of the X-axis as shown in 3 (a). Repeat the same argument with edges perpendicular to Y-axis and Z-axis to obtain Y-lines and Z-lines with label $Y = \pm 1, \pm 3, ..., \pm(2n - 1)$ and $Z = \pm 1, \pm 3, ..., \pm(2n - 1)$ respectively. See Figure 3 (b) and (c).

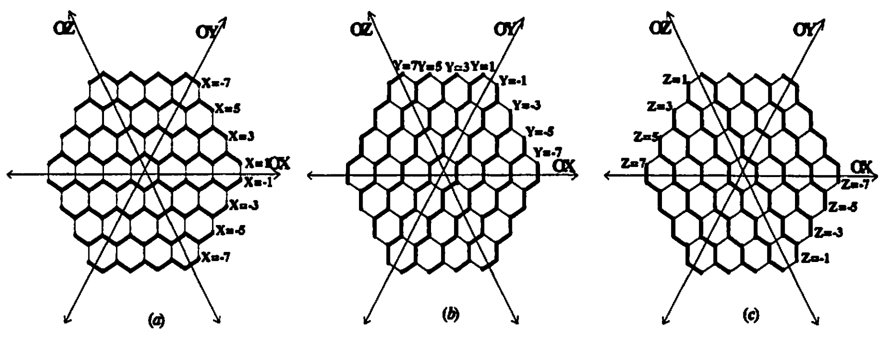

Figure 3: (a). X-lines (b). Y-lines (c). Z-lines

The coordinate of a vertex u in HM_n is the triple (i, j, k) when u is the point of intersection of the lines $X = i, Y = j$ and $Z = k$. The lines $X = i, Y = j$ and $Z = k$ divide the plane of the honeycomb mesh into six zones. Consider another vertex $v = (i', j', k')$ in the honeycomb mesh. Without loss of generality, let $i' > i$. Then the vertex v lies in zone I when $j' > j$ and $k' > k$, in zone II when $j' > j$ and $k' < k$ or in zone III when

$j' < j$ and $k' < k$. See Figure 4.

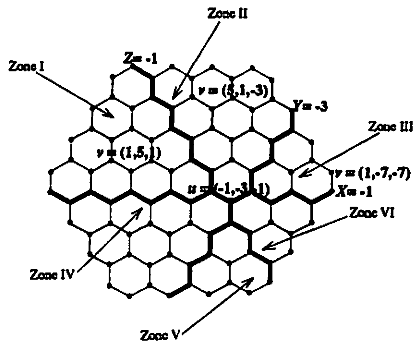

Figure 4: Six zones of honeycomb with vertices u and v

If v is in Zone I, the X-lines and Z-lines passing through u and v alone are sufficient to construct a parallelogram with u and v as the diagonally opposite vertices and the segments of the X and Z- lines as the boundary lines. The vertices in this parallelogram are addressed as 2-tuples using only the X-lines and the Z-lines. A similar same argument holds if v is in Zone II and Zone III. Let the parallelogram be denoted by $\lozenge_{uv}$. There are three types of parallelograms depending upon the position of v in the honeycomb mesh.

The parallelogram $\lozenge_{uv}$ bounded by X and Z- lines is called as an *α-parallelogram*, the parallelogram bounded by Y and Z-lines as a *β-parallelogram* and the one bounded by X and Y- lines as a *γ- parallelogram*. See Figure 5.

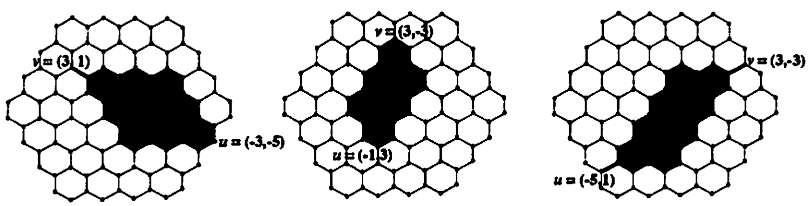

Figure 5: α-parallelogram, β-parallelogram, γ-parallelogram

Thus, given any two vertices u and v of HM_n we can form a honeycomb parallelogram with u and v as the diagonally opposite vertices. This honeycomb parallelogram plays a crucial role in finding the number of geodesics between the vertices u and v.

5 Main Results

The study of betweenness centrality of a vertex v in a graph G [11] and group betweenness centrality of a set $S \subseteq V(G)$ [5] has been an attractive area of research in social networks, computer networks and so on [1].

In the sequel let $d(s,t)$ denote the shortest distance between vertices s and t in a graph G, σ_{st} denote the number of geodesics between the vertices s and t and $\sigma_{st}(v)$ denote the number of geodesics between s and t passing through vertex v. Without loss of generality all results are discussed with respect to α-parallelograms.

Definition 1 *Betweenness centrality of a vertex v in a graph G is defined as* $BC(v) = \sum\limits_{s \neq v \neq t \in V} \delta_{st}(v)$ *where* $\delta_{st}(v) = \frac{\sigma_{st(v)}}{\sigma_{st}}$.

Definition 2 *Betweenness centrality of a set $S \subseteq V(G)$ is defined as* $BC(S) = \sum\limits_{s,t \in V \setminus S} \delta_{st}(S)$ *where* $\delta_{st}(S) = \frac{\sigma_{st(S)}}{\sigma_{st}}$ *where $\sigma_{st}(S)$ denotes the number of geodesics from s to t passing through at least one vertex of S.*

Lemma 1 *For vertices s and t in HM_n, all shortest paths between s and t lie in $\lozenge_{st}$.*

Proof. Stretching the boundary lines of the honeycomb parallelogram $\lozenge_{st}$ into straight lines yields a parallelogram in the plane. Since any parallelogram in the plane is convex, it contains all shorest paths between s and t. $\square$

The following lemma is evident from the addressing scheme.

Lemma 2 *Let $s = (i,j), t = (k,l)$. An α-parallelogram $\lozenge_{st}$ is composed of $\frac{|(k-i)|}{2}$ number of hexagons between two sucessive X-lines and $\frac{|(l-j)|}{2}$ number of hexagons between two sucessive Z-lines.*

Theorem 1 *Let $s = (i,j)$ and $t = (k,l)$. Then* $\sigma_{st} = \begin{pmatrix} (\frac{|k-i|}{2}) + (\frac{|l-j|}{2}) \\ (\frac{|l-j|}{2}) \end{pmatrix}$.

Proof. By lemma 2, there are $\frac{|(k-i)|}{2}$ number of hexagons between two sucessive X- lines and $\frac{|(l-j)|}{2}$ number of hexagons between two sucessive Z- lines. Hence the number of geodesics betweem s and t is

$$\sigma_{st} = \binom{(\frac{|k-i|}{2}) + (\frac{|l-j|}{2})}{(\frac{|l-j|}{2})}. \square$$

Remark 1 *For our convenience let us denote* $\sigma_{st} = \sigma_{(i,j),(k,l)}$ *as* $\sigma_{ij,kl}$.

Theorem 2 *Let* $s = (i,j), t = (k,l)$ *and* $v = (a,b)$. *Then*

$$\sigma_{st}(v) = \binom{(\frac{|a-i|}{2}) + (\frac{|b-j|}{2})}{(\frac{|b-j|}{2})} \binom{(\frac{|k-a|}{2}) + (\frac{|l-b|}{2})}{(\frac{|l-b|}{2})}$$

Proof. We have $\sigma_{ij,kl}(a,b) = \sigma_{ij,ab} \times \sigma_{ab,kl}$

$$= \binom{(\frac{|a-i|}{2}) + (\frac{|b-j|}{2})}{(\frac{|b-j|}{2})} \binom{(\frac{|k-a|}{2}) + (\frac{|l-b|}{2})}{(\frac{|l-b|}{2})}. \square$$

We need the following notations to determine the betweenness centrality measure of Honeycomb mesh HM_n.

Notation 3 *Let* v *be a vertex in a honeycomb mesh* HM_n. *The* X *and* Y*-lines through* v *partition* HM_n *into four sub honeycomb meshes* $A, B.C$ *and* D *and four paths* E_1, E_2, E_3 *and* E_4 *as shown in Figure.6.*

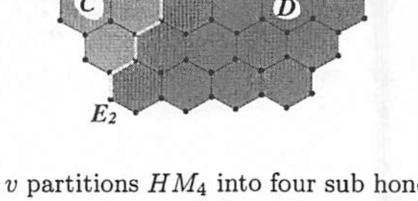

Figure 6: A vertex v partitions HM_4 into four sub honeycomb meshes and four paths

Notation 4 *Let* $X = \sum_{i=1}^{16} X_i$ *where* $X_1 = \{(s,t) : s \in A, t \in E_4\}$, $X_2 = \{(s,t) : s \in A, t \in D\}$, $X_3 = \{(s,t) : s \in A, t \in E_2\}$, $X_4 = \{(s,t) : s \in E_3, t \in D\}$, $X_5 = \{(s,t) : s \in E_3, t \in E_2\}$, $X_6 = \{(s,t) : s \in E_1, t \in E_4\}$, $X_7 = \{(s,t) : s \in E_1, t \in D\}$, $X_8 = \{(s,t) : s \in B, t \in E_2\}$, $X_9 = \{(s,t) : s \in B, t \in C\}$, $X_{10} = \{(s,t) : s \in B, t \in E_3\}$, $X_{11} = \{(s,t) : s \in E_4, t \in C\}, X_{12} = \{(s,t) : s \in E_4, t \in E_3\}, X_{13} = \{(s,t) : s \in E_1, t \in E_4\}, X_{14} = \{(s,t) : s \in E_1, t \in C\}, X_{15} = \{(s,t) : s \in E_1, t \in E_3\}, X_{16} = \{(s,t) : s \in E_1, t \in E_2\}$.

Theorem 5 *Let* $v = (a,b)$ *be any vertex in* HM_n. *Then the betweenness centrality of* v *is given by*

$$BC(v) = \sum_{s,t \in X} \frac{\binom{\frac{|b-j|}{2}+\frac{|a-i|}{2}}{\frac{|a-i|}{2}} \binom{\frac{|l-b|}{2}+\frac{|k-a|}{2}}{\frac{|k-a|}{2}}}{\binom{\frac{|l-j|}{2}+\frac{|k-i|}{2}}{\frac{|k-i|}{2}}}$$

where $s = (i,j)$ *and* $k = (k,l)$.

Proof. It is clear that $\sigma_{st}(v) = 0$ if $v \in V \setminus X$. On the other hand, $\sigma_{st}(v)$ contributes if $v \in X$.Hence

$$BC(v) = \sum_{s,t \in X} \frac{\binom{\frac{|b-j|}{2}+\frac{|a-i|}{2}}{\frac{|a-i|}{2}} \binom{\frac{|l-b|}{2}+\frac{|k-a|}{2}}{\frac{|k-a|}{2}}}{\binom{\frac{|l-j|}{2}+\frac{|k-i|}{2}}{\frac{|k-i|}{2}}}. \quad \square$$

Notation 6 *Let* $Z = (\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6)$ *be a hexagon in a honeycomb mesh* HM_n *where* $\alpha_1 = (a,b,c), \alpha_2 = (a,b,c-2), \alpha_3 = (a,b-2,c-2), \alpha_4 = (a-2,b-2,c-2), \alpha_5 = (a-2,b-2,c), \alpha_6 = (a-2,b,c)$. *It is bounded by* $X = a$, $X = a-2$, $Y = b, Y = b-2$, $Z = c$ *and* $Z = c-2$. *The* X*-lines and* Y*-lines partition* HM_n *into eight subhoneycomb meshes* $A, B, C, D, E_1, E_2, E_3$ *and* E_4 *as shown in the Figure 7.*

Lemma 3 *For any hexagon* Z *in the honeycomb mesh* HM_n, $\sigma_{st}(Z) = \sigma_{st}(\alpha_3) + \sigma_{st}(\alpha_6)$ *where* $s \in A, t \in D$.

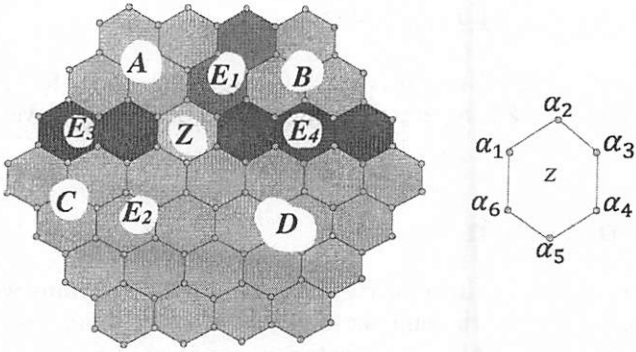

Figure 7: A hexagon Z partitions HM_4 into eight sub honeycomb meshes and a hexagon Z

Proof. The geodesics that pass through α_2 must also pass through α_3. Similarly the geodesics that pass through α_6 must pass through α_5. Therefore in finding the geodesics passing through Z, the contribution of geodesics passing through α_2 and α_5 is null. So the geodesics passing through α_1 are partitioned into paths that pass through α_3 and those that pass through α_6. Thus the geodesics passing through α_3 and α_6 contain all geodesics passing through α_1. Similarly all the geodesics passing through α_4 are nothing but paths passing through α_3 or α_6. So

$\sigma_{st}(Z) = (\sigma_{st}(\alpha_1) + \sigma_{st}(\alpha_3) + \sigma_{st}(\alpha_4) + \sigma_{st}(\alpha_6)) - \sigma_{st}(\alpha_1) - \sigma_{st}(\alpha_4)$.

Hence $\sigma_{st}(Z) = \sigma_{st}(\alpha_3) + \sigma_{st}(\alpha_6)$. $\square$

In what follows $S = \bigcup_{i=1}^{7} X_i$ and $T = \bigcup_{i=8}^{16} X_i$ form a partition of X.

Theorem 7 *Let Z be any hexagon in HM_n. Then the betweenness centrality of Z is given by*

$$BC(Z) = \sum_{s,t\in S} \frac{\left\{ \binom{\frac{|b-2-j|}{2}+\frac{|a-i|}{2}}{\frac{|a-i|}{2}} \binom{\frac{|l-b+2|}{2}+\frac{|k-a|}{2}}{\frac{|k-a|}{2}} \right\} + \left\{ \binom{\frac{|b-j|}{2}+\frac{|a-2-i|}{2}}{\frac{|a-2-i|}{2}} \binom{\frac{|l-b|}{2}+\frac{|k-a+2|}{2}}{\frac{|k-a+2|}{2}} \right\}}{\binom{\frac{|l-j|}{2}+\frac{|k-i|}{2}}{\frac{|k-i|}{2}}}$$

where $s = (i, j), t = (k, l)$ and $\{S, T\}$ is a partition of X.

Proof. For $s, t \in V \setminus X, \sigma_{st}(Z) = 0$. Since $s, t \in X_i$ for $1 \leq i \leq 16$ contribute to $BC(Z)$, repeated application of Lemma 3 proves the result. □

6 Conclusion

In this paper we have introduced honeycomb parallelograms which play a crucial role in finding the number of geodesics between any two vertices in a honeycomb mesh. Also exact betweenness centrality of a vertex and a set of vertices which induces a hexagon in a honeycomb mesh have been obtained.

References

[1] Bader, D., Madduri, K.A, "*Parallel algorithms for evaluating centrality indices in real world networks*", ICPP.Proc. 35^{th} International Conference on Parallel Processing, Columbus, OH, IEEE Computer Society, Los Alamitos, 2006.

[2] Brandes, U.,"*A faster algorithm for betweenness centrality*", Journal of Mathematical Sociology, 25(2), (2001) 163-177.

[3] Coffman, T., Grenblatt, S. and Marcus, S. "*Graph – based technologies for intelligence analysis*", Communications of the ACM,47, No. 3, (2004) 45-47.

[4] Eric D. Kolaczyk, David B. Chua, Marc Barthelemy "*Group betweenness and co – betweenness : Inter – related relations of coalition centrality*", an article in the press in the Journal of Social Neworks (2009).

[5] Everett, M. G., Borgatti, S.P., and Krackhardt, D.," *Ego – network Betweenness*", Paper presented at the 19^{th} International Conference on Socio Network Analysi (Sunbelt XIX), Charleston, South Carolina, 1999.

[6] Freeman, L.C., " *A set of measures of centrality based on betweenness*", Sociometry, 40, (1977) 35-41.