A Study On (a,d)-Antimagic Graphs Using Partition

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Abstract A connected graph $G(V,E)$ is said to be **(a,d)-antimagic** if there exist positive integers a and d and a bijection f: $E \to \{1,2,\ldots,|E|\}$ such that the induced mapping g_f: $V \to N$ defined by $g_f(v) = \Sigma_{e\in I(v)} f(e)$, where $I(v) = \{e\in E \,/\, e$ is incident to $v\}$, $v\in V$ is injective and $g_f(V) = \{a,a+d,a+2d,\ldots,a+(|V|-1)d\}$. In this paper, using partition, we prove that (i) the 1-sided infinite path P_1 is (1,2)-antimagic (ii) path P_{2n+1} is (n,1)-antimagic and (iii) (n+2,1)-antimagic labeling is the unique (a,d)-antimagic labeling of C_{2n+1} and graphs $K_1 + (K_1 \cup K_2)$, P_{2n} and C_{2n} are not (a,d)-antimagic. For $a,d\in N$, on (a,d)-antimagic graph G, we obtain a new relation, $a+(p-1)d \le \Delta(2q-\Delta+1)/2$. Using the results on (a,d)-antimagic labeling of C_{2n} and C_{2n+1}, we obtain results on the existence of (a,d)-arithmetic sequences of length 2n and 2n+1, respectively.

AMS Subject Classification: **05C78.**

Key words: antimagic labeling, antimagic graph, (a,d)-antimagic labeling, (a,d)-antimagic graph, partition, degree sequence.

1. Introduction

In 1990, Ringel [11] introduced the concept of antimagic graph. Each edge labeling f of a graph *G(V,E)* from 1 through $|E|$ induces a vertex labeling g_f where $g_f(v)$ is the sum of the labels of all edges that are incident upon vertex v, $v \in V$. Labeling f is called **antimagic** if and only if the values $g_f(v)$ are pair-wise distinct for all vertices v of *G*. Graph *G* is called **antimagic** if and only if it has an antimagic labeling. The main Illustration in the theory of antimagic graphs is the determination of all antimagic graphs. This Illustration still remains open. Ringel [11] conjectured that every connected graph *G* of order ≥ 3 is antimagic. In 1993, Bodendiek and Walther [4] introduced the concept of an **(a,d)-arithmetic antimagic** labeling.

Definition 1.1 [4] Let *G(V,E)* be a graph of order ≥ 3, and $a,d \in N$. A bijective mapping $f: E \to \{1,2,\ldots,|E|\}$ with induced mapping $g_f: V \to N$ defined by $g_f(v) = \Sigma_{e \in I(v)} f(e)$, where $I(v) = \{e \in E \mid e$ is incident to $v\}$, $v \in V$ is called **(a,d)-arithmetic antimagic labeling** or **(a,d)-antimagic labeling** if and only if $g_f(V)$ forms an arithmetic progression with initial value a and step width d. That is, $g_f(V) = \{a, a+d, a+2d, \ldots, a+(|V|-1)d\}$. See Figures 1 and 2. Figure 1 is (3,1)-antimagic graph G and Figure 2 is (4,1)-antimagic labeling of graph C_5.

G is called **(a, d)-arithmetic antimagic** or **(a, d)-antimagic** if and only if *G* admits an (a, d)-antimagic labeling. Clearly every (a,d)-antimagic graph is antimagic. The graph *G* given in Figure 1 is (3,1)-antimagic. The converse need not be true. For example, C_4 is antimagic as shown in Figure 3. But C_4 do not admit (a,d)-antimagic labeling for any pair $a,d \in N$.

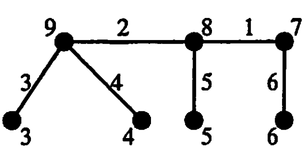

Fig. 1.

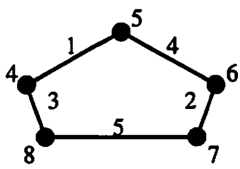

Fig. 2.

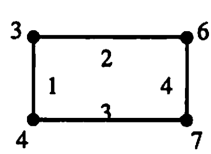

Fig. 3.

The **weight** w_f (v) (sometimes denoted as **w(v)**) of a vertex v in *V(G)* under an edge labeling f is the sum of values f(e) assigned to all edges e incident to v. Let *W* denote the set of all vertex weights of the graph *G*.

Bodendiek and Walther **[5,6]** proved the finiteness of two very interesting subsets of set of all (a,d)-antimagic parachutes. Baca and Hollander **[2]** characterized all (a,d)-antimagic graphs of prisms D_n = C_n x P_2 when n is even. They showed that when n is odd, the prism D_n are ((5n+5)/2,2)-antimagic. They also conjectured that prisms with odd cycles of length n, (n ≥ 7), are ((n+7)/2,4)-antimagic. Bodendiek and Walther **[7]** proved that the following graphs are not (a, d)-antimagic; even cycles; paths of even order, stars; $C_3^{(k)}$; $C_4^{(k)}$; for n ≥ 2 trees T_{2n+1} that have a vertex that is adjacent to three or more pendant vertices; n-ary trees with at least two layers when d = 1; $K_{3,3}$; the Petersen graph; and K_4. They also proved that P_{2k+1} is (k, 1)-antimagic; C_{2k+1} is (k+2,1)-antimagic; if a tree of odd order 2k+1 (k > 1) is (a,d)-antimagic, then d = 1 and a = k; if K_{4k} (k ≥ 2) is (a,d)-antimagic, then d is odd and d ≤ 2k(4k-3)+1; if K_{4k+2} is (a, d)-antimagic, then d is even and d ≤ (2k+1)(4k-1)+1; and if K_{2k+1} (k ≥ 2) is (a,d)-antimagic, then d ≤ (2k+1)(k-1). Nicholas, Somasundaram and Vilfred **[10]** gave condition for special types of caterpillars, spiders and complete bipartite graph $K_{m,n}$ to be (a,d)-antimagic and categorized (a,d)-antimagic labeling of unicyclic graphs and complete bipartite graphs.

Bodendiek and Walther [7] noted the following relation of (a,d)-antimagic graph.

Lemma 1.2 [7] Let G(V, E) be an (a, d)-antimagic graph with p ≥ 3 and q ≥ 2. Then a and d satisfy the following conditions:

(a) a,d∈ N are positive solutions of the linear Diophantine equation

$$(2a+(p-1)d)p = 2q(q+1)$$ and

(b) a ≥ δ(δ+1)/2 where δ denotes the minimum degree of G. ■

Definition 1.3 A partition of a non-negative integer n is a finite set of non-negative integers $d_1,d_2,\ldots,d_k$ whose sum is n.

Partition seems to be very simple but plays an important role in Combinatorics, Lie Theory, Representation Theory, Mathematical Physics and Theory of Special Functions. Euler, Ramanujan, Rademacher and Erdos revealed the beauty and uses of partition **[12]**. In **[13]**, using partition, Vilfred proved that graph $P_2 \times C_3$, the Cartesian Product of graphs P_2 and C_3 is (10,2)-antimagic and obtained all its 56 different possible (10,2)-antimagic labeling. In this paper, using partition, we prove that (i) the 1-sided infinite path P_1 is (1,2)-antimagic (ii) path P_{2n+1} is (n,1)-antimagic and (iii) (n+2,1)-antimagic labeling is the unique (a,d)-antimagic labeling of 2-regular graph of order 2n+1 and graphs $K_1+(K_1 \cup K_2)$, P_{2n} and 2-regular graph of even order are not (a,d)-antimagic. For any (a,d)-antimagic graph G, we derive a new relation, $a+(p-1)d \leq \Delta(2q-\Delta+1)/2$, $a,d \in N$. Using the results of (a,d)-antimagic labeling of C_{2n} and C_{2n+1}, we obtain the following.

1. For $n \geq 2$, sequence formed by taking sum of two distinct numbers out of 1,2,...,2n as an element will not admit (a,d)-arithmetic sequence of length 2n for any $a,d \in N$.

2. For $n \geq 1$, (n+2,1)-arithmetic sequence is the unique (a,d)-arithmetic sequence of length 2n+1 formed by taking sum of two distinct numbers out of 1,2,...,2n+1 as an element, $a,d \in N$.

For further readings on antimagic and (a,d)-antimagic labeling problems refer **[1-8,10,11,13]**. Motivation for this study is to use partition, a very simple technique, in the study of (a,d)-antimagic labeling of graphs. All graphs considered are simple graphs. For all basic ideas in graph theory, we follow **[9]**.

2. Main Results

The Diophantine equations in Lemma 1.2 are a necessary condition for the existence of (a,d)-antimagic graphs and we give one more relation in the following lemma.

Lemma 2.1 Let G be an (a,d)-antimagic graph and $p \geq 3$, $q \geq 2$, $a,d \in N$. Then, a and d satisfy the following conditions.

(a) $a,d \in N$ are positive solutions of the linear Diophantine equation

$$(2a+(p-1)d)p = 2q(q+1)$$ and

(b) $\delta(\delta+1)/2 \leq a$ and $L \leq \Delta(2q-\Delta+1)/2$ where $L = a+(p-1)d$ is the last term of the (a,d)-arithmetic antimagic series. That is the (a,d)-arithmetic antimagic series lies between $\delta(\delta+1)/2$ and $\Delta(2q-\Delta+1)/2$.

Proof

(a) We have, $\Sigma_{v\in V}\ g_f(v) = 2(\Sigma_{e\in E}\ f(e))$. This implies,

$$a+(a+d)+ \ldots +(a+(p-1)d) = 2(1+2+ \ldots +q)$$ which implies,

$$p(2a+(p-1)d)/2 = 2q(q+1)/2.$$ Hence the Theorem (a).

b) Since δ is the minimum degree of the graph G, at each vertex at least δ number of edges incident and hence the possible minimum value of the edges incident at a vertex is $\geq 1+2+\ldots+\delta$. This implies, for every vertex $v\in V(G)$, $g_f(v) = \Sigma_{e\in I(v)}\ f(e) \geq 1+2+\ldots+\delta = \delta(\delta +1)/2$. Thus, $a \geq \delta(\delta+1)/2$.

Similarly, the maximum possible induced vertex label is the sum of Δ distinct label of edges with possible maximum values. Thus, the possible maximum value of Δ edges is $q+(q-1)+(q-2)+\ldots+(q-\Delta+1) = \Delta(2q-\Delta+1)/2$. Since the maximum vertex value of the (a,d)-antimagic labeling of G is $L = a+(p-1)d$ and so $L = a+(p-1)d \leq \Delta(2q - \Delta+1)/2$. Hence the result. ■

Corollary 2.2 Let $G(V,E)$ be a k-regular (a,d)-antimagic graph with $p \geq 3$, $q \geq 2$, $a,d,k\in N$. Then, a,d and k satisfy the following conditions.

(a) $a,d\in N$ are positive solutions of the linear Diophantine equation

$$2a+(p-1)d = k(kp+2)/2$$ and

(b) $k(k+1)/2 \leq a$ and $L \leq k(kp-k+1)/2$ where $L = a+(p-1)d$ is the last term of the (a,d)-arithmetic antimagic series. That is the (a,d)-arithmetic antimagic series lies between $k(k+1)/2$ and $k(kp-k+1)/2$. ■

Theorem 2.3 Let G be a 2-regular graph of order p. If G is (a,d)-antimagic, then p is odd and $a = (p+3)/2$, $d = 1$, $a,d\in N$.

Proof If possible, let G be a 2-regular (a,d)-antimagic graph of order 2n for some $a,d,n\in N$. Then, equation (a) of Corollary 2.2 becomes

$2a+(2n-1)d = 4n+2$ which implies, d must be even. From equation (b) of Corollary 2.2, we get, $3 \le a$ and $L = a+(p-1)d = a+(2n-1)d \le 4n-1$. But (3,2)-antimagic sequence of length $p = 2n$ is $3,5,7,\ldots,3+2(2n-1) = 4n+1 > 4n-1$ and hence G is not a (3,2)-antimagic graph. In the above sequence, a and d take their minimum possible values and even in this case G is not (a,d)-antimagic for any a and d, $a,d \in N$. Hence, 2-regular (a,d)-antimagic graph of even order doesn't exist.

Now, let G be a 2-regular (a,d)-antimagic graph of order 2n+1, $n,a,d \in N$. Using equation (a) of Corollary 2.2, we get, $2a+2nd = 2(2n+1)+2$ which implies, $a = (2-d)n+2$, $a,d \in N$. This implies, $d = 1$ or 2. When $d = 2$, $a = 2$ which is not possible since $a \ge 3$. When $d=1$, $a = n+2 = (p+3)/2$, $p \in 2N+1$. Also, 2-regular ((p+3)/2,1)-antimagic graph of order p exists, see Illustration 2.13, $p \in 2N+1$. Hence the result. ■

Note 2.4 It is easy to prove that the graphs $C_3 \cup C_4$ and $3.C_3$ are not (a,d)-antimagic graphs even though each is a 2-regular odd order graph. It is not known whether disconnected 2-regular (a,d)-antimagic graph of odd order exists or not.

Theorem 2.5 Let G be a 3-regular graph of order p. If G is (a,d)-antimagic, then p is even and (i) $a = (7p+8)/4$ and $d = 1$, $p \in 4N$ or (ii) $a = 5(p+2)/4$ and $d = 2$, $p \in 4N+2$ or (iii) $a = (3p+12)/4$ and $d = 3$, $p \in 4N$ or iv) $a = (p+14)/4$ and $d = 4$, $p \in 4N+2$.

Proof Let G be a 3-regular (a,d)-antimagic graph of order p. Then, equations (b) and (a) of Corollary 2.2 become, $L=a+(p-1)d \le 3(3p-2)/2$; $6 \le a$ and $2a+(p-1)d = 3(3p+2)/2$ which implies, p must be even.

Let $p = 2n$, $n \in N$. In this case, $6 \le a$ and $L \le 3(3n-1)$. Equation (a) of Corollary 2.2 becomes, $2a+(2n-1)d = 9n+3$ which implies, $2a = 9n+3-(2n-1)d = (9-2d)n+3+d$, $a,n,d \in N$. The possible values of d are 1, 2, 3, 4. When $d = 1$, $a = (7n+4)/2$ and $L =a+(2n-1)d = (11n+2)/2 < 9n-3 = k(kp-k+1)/2$, $n \in 2N$. When $d = 2$, $a = 5(n+1)/2$ and $L = (13n+1)/2 < 9n-3$, $3 \le n \in 2N+1$. When $d = 3$, $a = (3n+6)/2$ and $L = 15n/2 < 9n -3$, $n \in 2N$. When $d = 4$, $a = (n+7)/2$ and $L = (17n-1)/2 < 9n-3$, $5 \le n$, $n \in 2N+1$. Hence the result. ■

Note 2.6 3-regular graph K_4 is not (a,d)-antimagic [8]. Graphs $K_{3,3}$ and $P_2 \times C_3$ are 3-regular graphs. Graph $K_{3,3}$ is not (a,d)-antimagic whereas $P_2 \times C_3$ is (10,2)-antimagic graph. Thus, the condition given in Theorem 2.5 need not be sufficient for the existence of (a,d)-antimagic 3-regular graph of even order.

Theorem 2.7 [Relation between (a,d)-antimagic and degree sequences]

Let G be a (p,q) graph with degree sequence $d_1,d_2,...,d_p$. Then, G is an (a,d)-antimagic if and only if there exists a set of p numbers, {a,a+d,...,a+(p-1)d}, formed by taking sum of exactly d_i number of numbers at a time out of 1,2,...,q, i = 1,2,...,p.

Proof Let G be an (a,d)-antimagic graph with degree sequence $d_1,d_2,...,d_p$. Then, the induced vertex labels of G are a,a+d,a+2d, ...,a+(p-1)d where 1,2,...,q are the edge labels of G. In G, each edge label is added exactly to its end vertices while considering the vertex labels of the graph and hence {a,a+d,...,a+(p-1)d} is the set of p numbers formed by taking sum of d_i number of numbers at a time out of 1,2,...,q, i = 1,2,...,p.

The converse part is obvious from the definition of (a,d)-antimagic graph. ■

The followings illustrate the application of Theorems 2.1 and 2.7.

Illustration 2.8 Show that the graph $G = K_1+(K_1 \cup K_2)$ is not (a,d)-antimagic, using partition technique, a,d∈N.

Solution Consider the graph $G = K_1+(K_1 \cup K_2)$. See Figure 4. Let 1,2,3,4 be the labels of the four edges e_1,e_2,e_3,e_4 of G. Using Theorem 2.1, possible (a,d)-antimagic sequences, if exist, are subsequences, each of length 4 of 1,2,...,9 = 4+3+2.

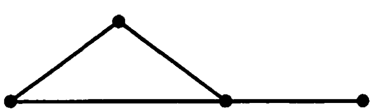

Fig.4. $G = K_1+(K_1 \cup K_2)$

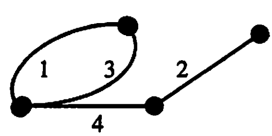

Fig.5

Thus, the possible (a,d)-antimagic sequences are

(a) 1,2,3,4 when a = 1, d = 1; (b) 2,3,4,5 when a = 2, d = 1; ...; (f) 6,7,8,9 when a = 6, d = 1; (g) 1,3,5,7 when a = 1, d = 2; (h) 2,4,6,8 when a = 2, d = 2 and (i) 3,5,7,9 when a = 3, d = 2.

Consider, each of the above (a,d)-antimagic sequences and their corresponding possible vertex labels. On each sequence, partitioning of vertex labels is done by one vertex label with three distinct parts, two vertex labels each of two distinct parts and one vertex label without partition, corresponding to the degree sequence of the graph 3,2,2,1.

Now, the label of the vertex with degree 3 should be $\geq$ 1+2+3 = 6 and so the sequences (a) and (b) are not possible. Also, one vertex label is without partition, corresponds to the pendant vertex and so its label should be $\leq$ 4. Hence cases (e) and (f) are not possible. In all other cases, we consider all possible schemes of partition one by one.

(c) In this case, the only possible scheme of partition is 6 = 1+2+3; 5 = 1+4; 4; 3 = 2+1.

(d) In this case, consider the following two scheme of partition:

(i) 7 = 4+2+1; 6 = 4+2; 5 = 1+4 or 2+3; 4 and

(ii) 7 = 4+3; 6 = 1+2+3; 5 = 4+1; 4.

(g) In this case, consider the following two scheme of partition:

(i) 7 = 1+2+4; 5 = 2+3; 3 = 1+2; 1 and

(ii) 7 = 1+2+4; 5 = 1+4; 3 = 1+2; 1.

(h) In this case, the scheme of partition is 8 = 4+3+1; 6 = 2+4; 4 = 1+3; 2 which is a (2,2)-antimagic labeling and the corresponding graph is given in Figure 5.

(i) In this case, the scheme of partition is 9=4+3+2; 7= 4+3; 5 = 2+3; 3.

Thus, in all these cases the vertex labels 1,2,3 and 4 are not occurring twice exactly, except case (h) which is also not possible for $K_1+(K_1 \cup K_2)$. Hence, (a,d)-arithmetic antimagic labeling does not exist for the graph $K_1+(K_1 \cup K_2)$, a,d$\in N$.

Hence, the graph $K_1+(K_1 \cup K_2)$ is not (a,d)-antimagic, a,d$\in N$. ■

Illustration 2.9 Show that C_4 is not (a,d)-antimagic, using partition technique.

Solution Let 1,2,3,4 be the edge labels of the four edges e_1, e_2, e_3, e_4 of C_4. The possible induced vertex labels are 3 (= 1+2 = minimum possible value), 4,5,6,7 (= 3+4 = maximum possible value). Therefore, 3,4,5,6 and 4,5,6,7 are the only possible (a,d)-antimagic sub-sequences, each of length 4, of the sequence 3,4,5,6,7. Now, consider the two cases separately. Since each vertex of C_4 is of degree two, in the partitioning of vertex labels each vertex label should be partitioned in to two different elements and the partitioned numbers (elements) should be from 1,2,3,4.

In the first case, the possible bipartition of the vertex labels are 6 = 2+4; 5 = 1+4 (= 2+3 is not possible for C_4 since 4 cannot occur as partitioned element in the other vertex labels); 4 = 1+3; 3 = 1+2. This is not possible since 3 occurs only once and 1 occurs three times. Hence, this case is not possible.

In the second case, the possible bipartition of the vertex labels are 7 = 3+4, 6 = 2+4. Here, 4 occurs twice and so its chance of occurrence is over and hence 5 = 2+3, 4 = 1+3. Thus, 3 occurs 3 times. Hence, this case is also not possible.

Hence, the graph C_4 is not (a,d)-antimagic for any $a,d \in N$. ■

Illustration 2.10 Show that the 1-sided infinite path P_1 is (1,2)-antimagic labeling, using partition technique.

Solution The starting and end vertices of the 1-sided infinite path P_1 is of degree one and all other vertices are of degree 2. If P_1 is (1,2)-antimagic labeling, then the vertex labels of P_1 are 1,3,5,7,.... Since 1 is a vertex label in P_1 implies, it is the label of a pendent vertex of P_1. Let the starting edge label be 1.

A partition scheme of the vertex labels is 1 = a, 3 = 1+2, 5 = 2+3, 7 = 3+4, 9 = 4+5 and so on. The last number of the (a,d)-antimagic is $a+(|V(G)|-1)d = 1+2(|V(G)|-1) = 2.|V(G)|-1$. Thus, the 1-sided infinite path P_1 is (1,2)-antimagic. See Figure 6. ■

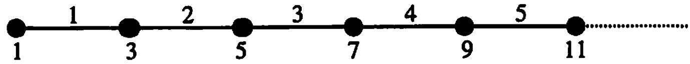

Fig. 6. (1,2)-antimagic labeling of 1-sided infinite path P_1.

Illustration 2.11 Show that every path P_{2n+1} is exactly (n,1)-antimagic, using partition technique, $n \geq 1$.

Solution For $n \geq 1$, assume that P_{2n+1} be (a,d)-antimagic, $a,d \in N$. Here, $p = 2n+1$ and $q = 2n$. The Diophantine equation (a) becomes, $2a(2n+1)+(2n+1)\cdot 2nd = 4n(2n+1)$. This implies, $a= n(2-d)$, $a,d,n \in N$. This equation has the unique solution, $d = 1$ and $a = n$. Thus, (n,1)-antimagic is the only possible (a,d)-antimagic labeling of P_{2n+1}, $n \geq 1$. The maximum value among the vertex label is $a+(p-1)d = 3n$. And so the sequence of induced vertex labels is n,n+1,...,3n-1,3n and the edge labels are 1,2,...,2n.

Consider the following bipartition scheme of the edge labels: $n = 0+n$; $n+1 = 0+(n+1)$; $n+2 = 1+(n+1)$; $n+3 = 1+(n+2)$; $n+4 =2+(n+2)$; ...; $3n-3 = (n-2)+(2n-1)$; $3n-2 = (n-1)+(2n-1)$; $(3n-1) = (n-1)+2n$; $3n = n+2n$. (n,1)-antimagic labeling of P_{2n+1} is given in Figure 7. ∎

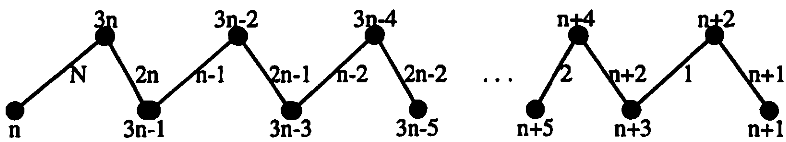

Fig. 7. (n,1)-antimagic labeling of P_{2n+1}.

Illustration 2.12 Show that for $n \geq 1$, path P_{2n} is not (a,d)-antimagic, $a,d \in N$.

Solution If possible, assume that for $n \geq 1$, path P_{2n} be an (a,d)-antimagic, for some $a,d \in N$. Then, the Diophantine equation (a) becomes, $2a(2n)+2n(2n-1)d = 4n(2n-1)$. This implies, $2a = (2n-1)(2-d)$. Hence, $d = 1$ is the only possible value of d and the corresponding value of a is $(2n-1)/2$, which is not a natural number.

Hence, the path P_{2n} is not an (a,d)-antimagic for $a,d,n \in N$. ∎

Illustration 2.13 Show that for $n \geq 1$, $(n+2,1)$-antimagic labeling is the unique (a,d)-antimagic labeling of C_{2n+1}, using partition technique.

Solution Assume for $n \geq 1$, C_{2n+1} be an (a,d)-antimagic for some $a,d \in N$. Applying the Diophantine equation (a) on C_{2n+1}, we get, $2a+2nd = 2(2(2n+1)+2)/2$. This implies, $a = n(2-d)+2$, $a,d,n \in N$. This equation has two solutions for d, namely, d = 2 and d = 1 and the corresponding values of a are a =n-2 and a = n+2, respectively. Thus, the following two cases arise.

Case 1 a = 2 and d = 2.

In this case, the vertex labels are 2,4,6,...,2(2n+1). This is not possible since 2 = 1+1 is the only bipartition of 2, which is not possible in C_{2n+1}. Hence, this case is not possible.

Case 2 a = n+2 and d = 1.

The possible induced vertex labels are n+2, n+3, . . . , (n+1)+(2n+1) = 3n+2 and by considering a bipartition of each of them, we have the following two sub cases.

Case 2.1 n is odd.

When n is odd, consider C_{2n+1} with the following partition scheme of the numbers (vertex labels of C_{2n+1}):

a = n+2 = (n+1)+1; n+3 = 1+(n+2); n+4 = (n+2)+2; . . . ;

2n+1 = (3n+1)/2+(n+1)/2; 2n+2 = (n+1)/2+3(n+1)/2; . . . ;

3n = 2n+n; 3n+1 = n+(2n+1); 3n+2 = (2n+1)+(n+1). Figures 8, 9 and 10 are the (3,1), (5,1) and (n+2,1)-antimagic labeling of graphs C_3, C_7 and C_{2n+1} when n is odd, respectively.

Case 2.2 n is even.

Consider C_{2n+1} when n is even.

The partition scheme of the numbers (vertex labels of C_{2n+1}) is

a = n+2 = 1+(n + 1); n+3 = 1+(n+2); n+4 = 2+(n+2); . . . ;

2n+1 = n/2+(3n+2)/2; 2n+2 = (3n+2)/2+(n+2)/2; . . . ;

3n = 2n+n; 3n+1 = n+(2n+1); 3n+2 = (2n+1)+(n+1).

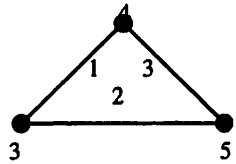

Fig. 8. C_3.

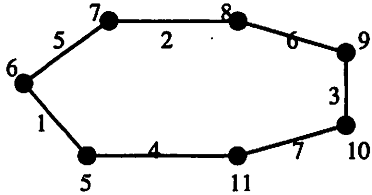

Fig. 9. C_7.

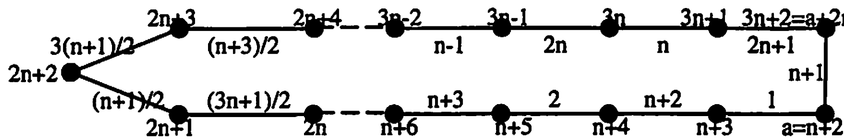

Fig. 10. C_{2n+1}, n is odd.

Figures 11, 12 and 13 show that the above partition scheme is a possible (a,d)-antimagic labeling in this case. ■

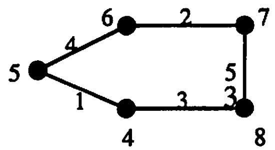

Fig. 11. C_5.

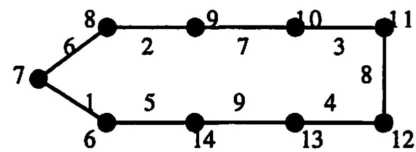

Fig. 12. C_9.

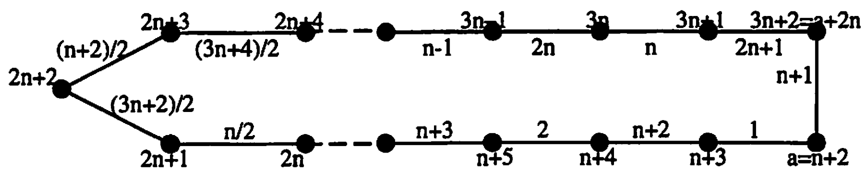

Fig. 13. C_{2n+1}, n is even.

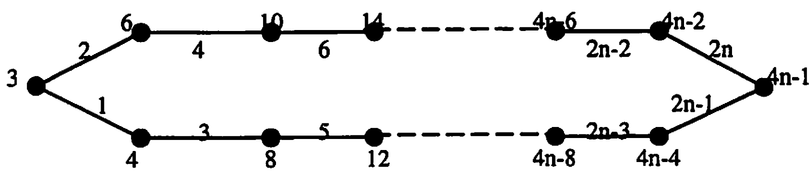

Fig. 14. Antimagic labeling of C_{2n}.

For $n \geq 2$, C_{2n} is antimagic, see Figure 14, but it is not (a,d)-antimagic, see Illustration 2.14, $a,d \in N$.

Illustration 2.14 Show that for $n \geq 2$, the graph C_{2n} is not (a,d)-antimagic, $a,d \in N$.

Solution If possible, let C_{2n} be an (a,d)-antimagic for some $a,d \in N$ and $n \geq 2$. Applying the Diophantine equation (a) on C_{2n}, we get,

$$2a+(2n-1)d = 2(2n+1) \quad --- (1)$$

Also, $a \geq 1+2 = 3$ and $a+(p-1)d = a+(2n-1)d \leq 2q-1 = 4n-1$ which implies, $2a+2(2n-1)d \leq 8n-2$. This implies, $4n+2+(2n-1)d \leq 8n-2$, using equation (1). This implies, $(2n-1)d \leq 4(n-1)$, $n \geq 2$, $d,n \in N$. The only possible solution of this relation is $d = 1$ and the corresponding value of a is obtained from equation (a) as $2a = 2n+3$ which implies, $a = (2n+3)/2$ which is not a natural number. Hence, C_{2n} is not (a,d)-antimagic for any $a,d,n \in N$ and $n \geq 2$. ■

Note 2.15 It is noted that the proof given in Results 2.13 and 2.14 are true in the case of 2-regular graphs of order 2n+1 and 2n, respectively, $n \in N$. Thus, in general, we get the following results.

Illustration 2.16 Show that for $n \geq 1$, (n+2,1)-antimagic labeling is the unique (a,d)-antimagic labeling of 2-regular graph of order 2n+1. That is (n+2,1)-antimagic labeling is the unique (a,d)-antimagic labeling, if exists, of disjoint union of cycles whose total number of vertices is 2n+1, $a,d,n \in N$. ■

Illustration 2.17 Show that for $n \geq 2$, 2-regular graph of order 2n is not (a,d)-antimagic, $a,d,n \in N$. That is disjoint union of cycles whose order is 2n is not (a,d)-antimagic, $a,d,n \in N$ and $n \geq 2$. ■

3. Combinatorial Interpretation of results on Cycles

It is clear that any (a,d)-antimagic graph is antimagic but the converse need not be true. Also, for $n \geq 2$, 2-regular graph of order 2n is not (a,d)-antimagic, $a,d \in N$ whereas for $n \geq 1$, (n+2,1)-antimagic labeling is the unique (a,d)-antimagic labeling, if exists, of 2-regular graph of order 2n+1. The above results have interesting combinatorial interpretation.

Definition 3.1 For a,d$\in N$, the sequence a, a+d, a+2d,..., a+(n-1)d is called an **(a,d)-arithmetic sequence of length n.**

Theorem 3.2 For n $\geq$ 2, the sequence formed by taking sum of two distinct numbers out of 1,2,...,2n as an element will not admit any (a,d)-arithmetic sequence of length 2n, a,d$\in N$.

Proof It is enough to prove that for n $\geq$ 2, 2-regular graph of order 2n is not (a,d)-antimagic, a,d,n$\in N$. The Theorem follows from Illustration 2.17. ■

Theorem 3.3 For n $\geq$ 1, (n+2,1)-arithmetic sequence is the unique (a,d)-arithmetic sequence of length 2n+1 formed by taking sum of two distinct numbers out of 1,2,...,2n+1 as an element, a,d$\in N$.

Proof It is enough to prove that for n $\geq$ 1, (n+2,1)-antimagic labeling is the unique (a,d)-antimagic labeling of 2-regular graph of order 2n+1. The Theorem follows from Problems 2.16 and 2.13. Labeling given in the solution of Illustration 2.13 establishes the existence of such labeling. ■

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