Graphs with Minimum Spanner $\varsigma(G) \geq 2\rho - 1$

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Abstract

The parameter t of a tree t-spanner of a graph is always bounded by 2λ where λ is the diameter of the graph. In this paper we establish a sufficient condition for graphs to have the minimum spanner at least $2\rho - 1$ where ρ is the radius. We also obtain a characterization for tree 3-spanner admissible chordal graphs in terms of tree 3-spanner admissibility of certain subgraphs.

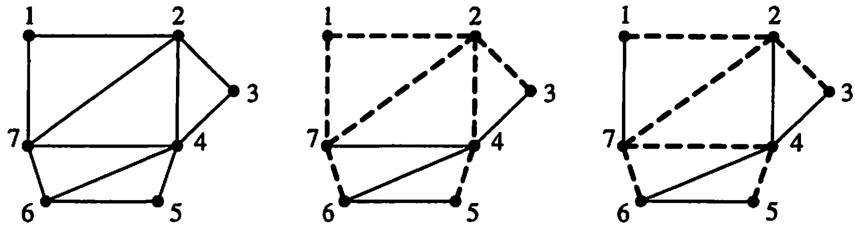
Keywords: spanning subgraph, tree t-spanner, Petersen graph, chordal graph, split graph.

1. Introduction

An *interconnection network* consists of a set of processors, each with a local memory, and a set of bidirectional links that serve for the exchange of data between processors. A convenient representation of an interconnection network is by an undirected (in some cases directed) graph $G = (V, E)$ where each processor is a vertex in V and two vertices are connected by an edge if and only if there is a communication link (bidirectional for undirected and unidirectional for directed graphs) between processors[13]. We will use the term interconnection network and graph interchangeably.

Design of interconnection networks is an integral part of parallel processing or distributed systems. There are a large number of topological choices for interconnection networks. If a network has an expensive topology, a sparse less expensive spanner can be substituted, while retaining a similar network structure with a slight increase in communication costs.

Given a simple connected graph G, a spanning subgraph H of G is a t-spanner of G if for every $u, v \in V(G)$, the distance between u and v in H is at most t times their distance in G. Peleg and Schaffer [12] proved that a spanning subgraph H of G is a t-spanner of G if and only if for edge $(x, y) \in E(G)$, the distance between x and y in H is at most t. A t-spanner is minimum if it contains a minimum number of edges among all t-spanners of G. The minimum t-spanner problem is to find a t-spanner with the minimum number of edges for a given graph and a given t [5]. A tree t-spanner T in a graph G is a spanning tree of G such that the distance between every pair of vertices in T is at most t times their distance in G. See Figure 1. A graph G is tree t-spanner admissible if it contains a tree t-spanner. The tree t-spanner admissible problem is to determine the existence of a tree t-spanner in a given graph [3, 4]. The minimum tree spanner problem is to find a tree t-spanner with the minimum t for a given graph [3, 8, 9].



(a) (b) (c)

Figure 1: (a). A graph G. (b). A 4-spanner. (c). A tree 4-spanner

Let G be a graph and v be a vertex of G. The *eccentricity* of the vertex v is the maximum distance from v to any vertex. That is, $e(v) = \max\{d(v, w): w \text{ in } V(G)\}$.

The minimum eccentricity among the vertices of G is termed as radius ρ and the maximum eccentricity among the vertices of G the diameter λ. In other words

$$\rho(G) = \min\{e(v): v \text{ in } V(G)\}$$

$$\lambda(G) = \max\{e(v): v \text{ in } V(G)\}.$$

Let G be a graph with diameter λ. A vertex v of G is said to be diametrically opposite to a vertex u of G, if $d_G(u, v) = \lambda$. A graph G is said to be diametrically uniform if every vertex of G has at least one diametrically opposite vertex. The set of diametrically opposite vertices of a vertex x in G is denoted by $D(x)$.

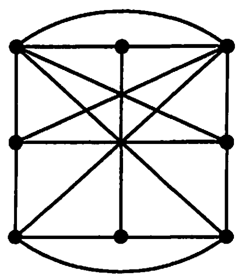


Figure 2: A diametrically uniform graph with diameter 2

In this paper, we deviate from the notion of spanners defined in the literature. The term spanner in the literature is a spanning subgraph whereas our concept of spanner is a number.

A spanner $\varsigma(T, G)$ of a spanning tree T of G is defined as $\varsigma(T, G) = \max\{d_T(u, v): (u, v) \text{ is an edge of } G\}$. The minimum spanner $\varsigma(G)$ of G is defined as $\varsigma(G) = \min\{\varsigma(T, G): T \text{ is a spanning tree of } G\}$. A spanning tree T

is called a minimum tree spanner, if $\varsigma(T, G) = \varsigma(G)$. Equivalently T is a minimum tree spanner if $\varsigma(T, G) \leq \varsigma(T', G)$, for all spanning trees T' of G [11].

In this paper, we establish a sufficient condition for graphs to have the minimum spanner at least $2\rho - 1$ where ρ is the radius.

2. Graphs with minimum spanner at least $2\rho - 1$

Let G be a graph and let $u \in V(G)$. Let $R(u) = \{v \in V: d(u, v) = \rho\}$ where ρ is the radius of G. We establish a sufficient condition for graphs to have minimum spanner at least $2\rho - 1$.

Theorem 1: Let $G = (V, E)$ be a graph. If for every edge (x, y) in E and for every vertex x^* of $R(x)$, there exists a vertex y^* of $R(y)$ such that (x^*, y^*) is an edge of G, then $\varsigma(G) \geq 2\rho - 1$.

Proof. Assume the contrary. Let there be a spanning tree T such that

$$d_T(x, y) < 2\rho - 1, \text{ for every edge } (x, y) \text{ in } E. \qquad (1)$$

Without loss of generality, let T be a rooted tree. Given a vertex α and a member α^* of $R(\alpha)$ such that α^* is a descendant of α in T, we claim that the vertex α has a child β such that the subtree rooted at β contains α^* as well as a member β^* of $R(\beta)$. See Figure 3.

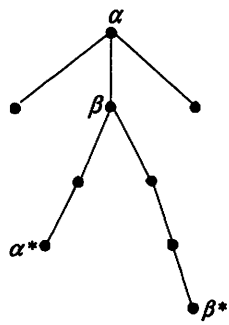


Figure 3: Subtree rooted at β contains α^* and β^*

Let β be a child of α such that the subtree rooted at β contains α^*. This is possible since $\rho \geq 1$. Since $(\alpha, \beta) \in E$, by hypothesis of the theorem, there exists a vertex β^* of $R(\beta)$ such that $(\alpha^*, \beta^*) \in E$. Now it is enough to prove that α^* and β^* are in the same subtree rooted at β. Suppose that our claim is false.

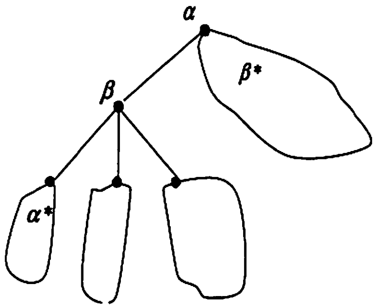


Figure 4: α^* and β^* are in the different subtrees rooted at β

Then α^* and β^* lie in two different components of $T - \alpha$ and hence $d_T(\alpha^*, \beta^*) \geq 2\rho - 1$. See Figure 4. This is a contradiction to condition (1) since $(\alpha^*, \beta^*) \in E$. This proves our claim.

We start at the root of T and traverse T in a DFS order. Let γ denote the root of T. Once a vertex x of T is visited, a child y of x is identified and visited inductively. Now let us start from the root γ of T. Let $\gamma^* \in R(\gamma)$.

Given γ and a member γ^* of $R(\gamma)$ in the subtree rooted at γ, we can find a child δ of γ such that the subtree rooted at δ contains γ^* as well as a member δ^* of $R(\delta)$. From γ, it traverses to δ. The DFS traversal starting at γ visits δ, a child of γ.

Inductively let x be the last visited vertex and x^* be a member of $R(x)$ which is a descendant of x. As we have shown above, there exists a child y of x such that the subtree rooted at y contains x^* as well as a member y^* of $R(y)$. Hence the DFS traversal never reaches a leaf and does not terminate which is not possible in a finite tree. Thus $\varsigma(G) \geq 2\rho - 1$.

Remark 1: When the radius equals the diameter the graph reduces to a diametrically uniform graph and thus we have the following result.

Corollary 1 [11]: Let G be a diametrically uniform graph with diameter $\lambda > 1$. Given an edge (x, y) in $E(G)$, if for every vertex x^* of $D(x)$ there exists a vertex y^* of $D(y)$ such that (x^*, y^*) is an edge of G, then $\zeta(G) \geq 2\lambda - 1$, where $D(x)$ is the set of vertices diametrically opposite to x.

3. Minimum Spanner of Odd Petersen Graphs

A generalized Petersen graph $P(n, m)$, $n \geq 3$, $1 \leq m \leq \left\lfloor \frac{n-1}{2} \right\rfloor$, consists of an outer n – cycle $u_1u_2\ldots u_n$, a set of n spokes (u_i, v_i), $1 \leq i \leq n$ and n inner edges

(v_i, v_{i+m}) with indices taken modulo n. For convenience, $u_1, u_2, \ldots, u_n$ are represented by $1, 2, \ldots, n$ and $v_1, v_2, \ldots, v_n$ by $n + 1, n + 2, \ldots, 2n$ respectively. In this paper, we consider Petersen graphs with $m = 2$ and call a generalized Petersen graph $P(n, 2)$ simply a Petersen graph.

The diameter of Petersen graph $P(n, 2)$ is given by $\lambda = \left\lfloor \frac{(n-6)}{4} \right\rfloor + 4$, $n \geq 8$.

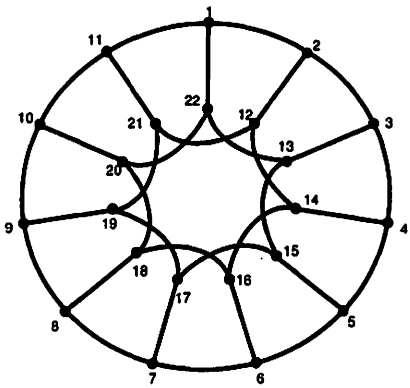


Figure 5: A Petersen graph P(11,2)

Proposition 1 [11]: The Petersen graph $P(2n, 2)$ is diametrically uniform.

Proposition 2 [11]: The Petersen graph $P(2n + 1, 2)$, $n \geq 4$, is not diametrically uniform and the radius is given by $\rho = \left\lfloor \frac{(n-3)}{2} \right\rfloor + 3$.

Proposition 3: Let G be $P(2n + 1, 2)$, $n \geq 4$. Then, for every edge (x, y) in E and for every vertex x^* of $R(x)$, there exists a vertex y^* of $R(y)$ such that (x^*, y^*) is an edge of G.

Theorem 2: Let G be $P(4n + 3, 2)$, $n \geq 2$. $\varsigma(G) = 2\rho - 1$.

Proof. Theorem 1 and Proposition 3 yield $\varsigma(P(4n + 3, 2)) \geq 2\rho - 1$. The Breadth-First Search Algorithm to draw the BFS tree rooted at an inner cycle vertex of $P(4n + 3, 2)$, results in a spanning tree with $\varsigma(P(4n + 3, 2)) = 2\rho - 1$.

Open Problem: Let G be $P(4n + 1, 2)$, $n \geq 2$. $\varsigma(G) = 2\rho$.

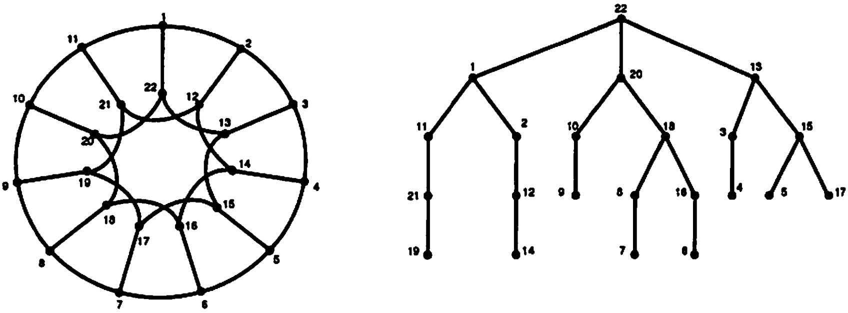


Figure 6: An example to illustrate Theorem 2

4. Conclusion

We have established a sufficient condition for graphs to have minimum spanner at least $2\rho - 1$. The minimum spanner of odd Petersen graph has been derived. A future direction of research is to identify more classes of graphs with minimum spanner $2\rho - 1$.

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