# Equi-Toughness Partitions of Graphs

R. Sundareswaran and V.Swaminathan
Ramanujan Research Center in Mathematics,
Saraswathi Narayanan College, Madurai
Email: sulanesri@yahoo.com

**Abstract**

Let G = (V, E) be a simple graph. Let S be a subset of V (G). The toughness value of S denoted by $T_S$ is defined as $\frac{|S|}{\omega(G-S)}$, where $\omega(G-S)$ denotes the number of components in G - S. If S = V, then $\omega(G-S)$ is taken to be 1 and hence $T_{V(G)} = |V(G)|$. A partition of V (G) into subsets $V_1, V_2, \ldots V_t$ such that $T_{V_i}$, $1 \leq i \leq t$ is a constant is called an equi-toughness partition of G. The maximum cardinality of such a partition is called equi-toughness partition number of G and is denoted by ET (G). The existence of ET-partition is guaranteed. In this paper, a study of this new parameter is initiated.

**Keywords:** Toughness, equi-toughness partition

## 1. Introduction

Let G = (V, E) be a simple graph. Then the toughness t(G) of G is the minimum of $\frac{|S|}{\omega(G-S)}$ taken over all sets S of vertices such that $\omega(G-S) \geq 2$, where $\omega(G)$ denote the number of components of G. A subset S of V(G) for which the minimum is achieved is called a tough set. The parameter toughness was introduced by Chv´atal [3]. Though much of the research has focused on the relationship between toughness and hamiltonicity, some general results were derived by Pippert and Goddard [5] and Swart [4].

In the case of toughness of a graph, subsets S of V with $\omega(G-S)\geq 2$ are alone considered. For any subset S of V we can associate a value $\frac{|S|}{\omega(G-S)}$. If $\omega(G-S)=1$, then this value will be |S|. We call this value as the toughness value of S and we denote this by $T_S$. If $S = V$, then $\omega(G-S)$ is taken to be 1 and hence $T_{V(G)} = |V(G)|$. A partition of V (G) into subsets $V_1, V_2, \ldots, V_t$ such that $T_{V_i}$, $1 \leq i \leq t$ is a constant is called an equi-toughness partition of G. The maximum cardinality of such a partition is called equi-toughness partition number of G and is denoted by ET (G). A study of this equi-toughness partition is made in this paper.

## 2.Equi-toughness partition

**Definition 2.1:** Let G = (V, E) be a simple graph. Let S be a subset of V (G). The toughness value of S denoted by $T_S$ is defined as $\frac{|S|}{\omega(G-S)}$, where $\omega(G-S)$ denotes the number of components in G - S. If S = V, then $\omega(G - S)$ is taken to be 1.

**Definition 2.2:** Let G = (V, E) be a simple graph. A partition of V (G) into subsets $V_1$, $V_2$, ... $V_t$ such that $T_{V_i}$, $1 \leq i \leq t$ is a constant is called an equi-toughness partition of G. The maximum cardinality of such a partition is called equi-toughness partition number of G and is denoted by ET (G). Since V (G) itself is an equi-toughness partition of G, the existence of an equi-toughness partition in any graph is guaranteed.

**Observation 2.3:** For any graph G, $1 \leq ET(G) \leq n$ and the bounds are attained as seen in Theorem 2.4 and Proposition 2.6 below.

**Theorem 2.4:** Let G be a nontrivial connected graph. Then ET (G) = n if and only if G has no cut vertex.

*Proof.*
Suppose G is a nontrivial connected graph without cut vertex. Then ET (G) = n.
Conversely, suppose ET (G) = n. Then $T_u$ is constant for every vertex $u \in V$ (G). Since G is a nontrivial connected graph, G has at least two vertices which are not cut vertices. For such a vertex say u, $T_u = 1$. Hence $T_u = 1$ for all $u \in V$ (G). If G has cut vertex say v, then $T_v = \frac{1}{\omega(G-v)} < \frac{1}{2} < 1$, a contradiction. Therefore, G has no cut vertices.

**Proposition 2.5:** $ET(K_n) = n$; $ET(K_{m,n}) = m + n$; $ET(W_n) = n$; $ET(C_n) = n$.

**Proposition 2.6:**

$$ET(K_{1,n}) = \begin{cases} \frac{n+3}{2}, & \text{if n is odd} \\ 1, & \text{otherwise} \end{cases}$$

*Proof.*

**Case (i):** Let n + 1 be odd. Let $n = 2\ell$.

Suppose $ET(K_{1,n}) > 1$. Let $\{V_1, V_2, \ldots, V_s\}$ be a maximum equi-toughness partition of $K_{1,n}$. Without loss of generality, let the center of the star belong to $V_1$. Let $|V_1| = k$ and $T_{V_1} = t$. Therefore, $\frac{k}{n+1-k} = t$, (since $k < n+1$, left hand side is finite). Therefore, $k = \frac{(n+1)t}{t+1}$ and hence $k - 1 = \frac{(n-1)}{t+1}$.

Since $T_{V_2} = T_{V_3} = \ldots = T_{V_s} = t$, $|V_2| = |V_3| = \ldots = |V_s| = t$.

Thus, $|V_2| + |V_3| + \ldots + |V_s| = 2\ell - (k-1) = 2\ell - \frac{(n-1)}{t+1}$.

Therefore, $2\ell - \frac{(n-1)}{t+1} = t(s-1)$, $s \geq 2$.

Therefore, $2\ell t + 1 = t(t+1)(s-1)$, a contradiction, since left hand side is odd and right hand side is even. Therefore, $ET(K_{1,n}) = 1$.

**Case (ii):** Suppose n +1 is even.

Let $V(K_{1,n}) = \{u, v_1, v_2, \ldots, v_n\}$, where u is the center of the star $K_{1,n}$.

Let $V_1 = \{u, v_1, v_2, \ldots, v_{\frac{n-1}{2}}\}$, $V_2 = \{v_{\frac{n+1}{2}}\}$, $V_3 = \{v_{\frac{n+3}{2}}\}$, ..., $V_{\frac{n+3}{2}} = \{v_n\}$. Then, $T_{V_1} = T_{V_2} = \ldots = T_{V_{\frac{n+3}{2}}} = 1$. Thus, $ET(K_{1,n}) \geq \frac{n+3}{2}$. Suppose $ET(K_{1,n}) = \ell > \frac{n+3}{2}$. Let $\{V_1, V_2, \ldots, V_\ell\}$ be a maximum equi-toughness partition of $K_{1,n}$. Let $u \in V_1$. Let $T_{V_1} = t$ (say). Let $|V_1| = k$. Therefore, $\frac{k}{n+1-k} = t$ implies $k - 1 = \frac{n-1}{t+1}$.

Since $T_{V_2} = T_{V_3} = \ldots = T_{V_s} = t$, $|V_2| = |V_3| = \ldots = |V_\ell| = n - \frac{n-1}{t+1}$. Therefore, $n - \frac{n-1}{t+1} = t(\ell - 1)$, $\ell \geq 2$ implies $\ell = \frac{nt+1}{t(t+1)} + 1$. Therefore, $\frac{nt+1}{t(t+1)} + 1 > \frac{n+3}{2}$ implies $(n+1)t^2 + (1-n)t - 2 < 0$. $(n+1)t^2 + (1-n)t - 2 = 0$ gives $t = 1$ or $\frac{-2}{n+1}$. Therefore, $(n+1)t^2\ (1-n)t - 2 < 0$ implies t lies between 1 and $\frac{-2}{n+1}$. Therefore $t < 1$, a contradiction, since $\ell > \frac{n+3}{2}$ implies $t \geq 1$. Thus, $\ell = \frac{n+3}{2}$.

Hence $ET(K_{1,n}) = \frac{n+3}{2}$.

**Proposition 2.7:**

$$EI(P_n) = \begin{cases} 1 \text{ if } n = 3 \\ \left\lfloor \frac{n}{2} \right\rfloor + 1 \text{ if } n \geq 4 \end{cases}$$

*Proof.*

It can be easily seen that ET $(P_3)$ =1. Let $n \geq 4$. Let $V(P_n) = \{v_1, v_2, \ldots, v_n\}$. If n is odd, then $\{\{v_1, v_{n-1}\}, \{v_2, v_3\}, \{v_4, v_5\}, \ldots, \{v_n\}\} = \{V_1, V_2, \ldots, V_k\}$, where $k = \left\lfloor \frac{n}{2} \right\rfloor + 1$ is an ET-partition with $T_{V_i} = 1$, for all i, $1 \leq i \leq k$. If n is even, then $\{\{v_1\}, \{v_2, v_3\}, \ldots, \{v_{n-2}, v_{n-1}\}, \{v_n\}\} = \{V_1, V_2, \ldots, V_k\}$, where $k = \left\lfloor \frac{n}{2} \right\rfloor + 1$ is a maximum ET-partition with $T_{V_i} = 1$, for all i, $1 \leq i \leq k$.

Therefore, EI $(P_n) \geq \left\lfloor \frac{n}{2} \right\rfloor + 1$. It can be easily seen that ET $(P_n) \leq \left\lfloor \frac{n}{2} \right\rfloor + 1$ if $n \geq 4$. Hence the result.

**Proposition 2.8:** Let G be a connected graph of order n with $\kappa(G) \geq 2$ and $G^+$, the carona of G. Then $ET(G^+) = |V(G)| + 1$.

*Proof.*

Let $V(G) = \{u_1, u_2, \ldots, u_n\}$ and let $v_i$ be the pendent adjacent to $u_i$ in $G^+$, $1 \leq i \leq n$. Consider the partition $\pi = \{\{v_1\},\{v_2\}, \ldots, \{v_n\}, V(G)\}$. $\pi$ is an equi-toughness partition since $T_{V_i} = 1$ and $T_{V(G)} = 1$. Therefore, ET $(G^+) \geq n + 1$. Suppose ET $(G^+) = t \geq n + 2$. Let $\pi = \{V_1, V_2, \ldots, V_t\}$ be a maximum equi-toughness partition of $G^+$. If there are $(t - 3)$ sets in $\pi$ with cardinality at least 2, then $|V(G^+)| = 2n \geq (t - 3)\,2 + 3 = 2t - 3 \geq 2n + 1$, a contradiction. Therefore there are at most $(t - 4)$ sets in $\pi$ of cardinality at least 2. Therefore there are at least 4 sets in $\pi$ each having cardinality 1. Since $T_u$ is 1, if u is a pendent and $\frac{1}{2}$ if u is a support, a support and a pendent cannot appear as singletons in the partition. Therefore, the singleton sets are either formed by pendents or by supports.

**Case (1):** Suppose the singleton sets are formed by pendents. The number of pendents available for the sets in the partition $\pi$ with the cardinality at least 2 is at most $n - 4$. Also, $T_{V_i} = 1$ for all i, $1 \leq i \leq t$. $T_{V_i} = 1$ if and only if either $V_i = \{v\}$, where v is a pendent or there are k supports and a single pendent in $V_i$ and the single pendent is not adjacent to any of the supports in $V_i$ or $V_i = V(G)$. If $V_i = V(G)$, then $V_j$, $j \neq i$ are all singletons and each of them is a pendent. Therefore $t = n + 1$, a contradiction. Thus, no $V_i$ is $V(G)$. If every $V_i$ is a singleton consisting of a pendent, then $t = n$, a contradiction. Therefore some $V_i$ contains $k + 1$ vertices where k of the elements are supports and the remaining is a pendent which is not adjacent to the supports $(k \geq 1)$. Since there are n supports and at most $n - 4$ pendents available for the sets containing $k + 1$ elements $(k \geq 1)$, one of the sets must contain at least 4 supports and the total number of sets is at most n - 4. Therefore $t \leq n$, a contradiction.

**Case (2):** Suppose the singleton sets are formed by supports. Then $T_{V_i} = \frac{1}{2}$, for all $i$, $1 \leq i \leq k$. Therefore any $V_i$ can not be made up of pendents only. Thus, either $V_i$ is a singleton containing a support or $V_i$ has both supports and pendents. Thus, with at most n - 4 supports available, we can make at most n - 4 sets. Therefore $t \leq n$, a contradiction.

**Definition 2.9:** A double star denoted by $D_{r,s}$ is formed by joining the centers of two stars $K_{1,r}$ and $K_{1,s}$.

**Proposition 2.10:** For any tree T of order $n \geq 4$, $ET(T) \leq \left\lfloor \frac{n}{2} \right\rfloor + 1$.

*Proof.*

On the same lines as in Proposition (2.4) with the observation that $T_u = \frac{1}{\deg(u)}$, for any support u.

**Remark 2.11:** The bound is reached in $P_n$ ($n \geq 4$), $D_{r,s}$ where r and s are of the same parity and Binary trees.

**Proposition 2.12:**

$$EI(D_{r,s}) = \begin{cases} \frac{|V(D_{r,s})|}{2} + 1 & \text{if r,s are of the same parity} \\ \frac{|V(D_{r,s})| + 3}{2} & \text{if r,s are not of the same parity} \end{cases}$$

*Proof.*

Let V $(D_{r,s}) = \{u, v, v_1, v_2, \ldots, v_s, u_1, u_2, \ldots, u_r\}$ where u, v are the centers of the double star $D_{r,s}$ and $u_i$, $1 \leq i \leq r$ are the pendent vertices adjacent with u and $v_j$, $1 \leq j \leq s$ are the pendent vertices adjacent with v.

**Case (A):** Let r, s be even.

**Case (i):** Suppose r = s.

Then $\{\{u, v, v_1, v_2, \ldots, v_{s-1}\}, \{u_1\}, \{u_2\}, \ldots, \{u_r\}, \{v_s\}\} = \{V_1, V_2, \ldots, V_k\}$, where $k = s + 2 = \frac{r+s}{2} + 2 = \frac{r+s+2}{2} + 1$ is an ET-partition with $T_{V_i} = 1$, for all i, $1 \leq i \leq k$. Therefore, ET $(D_{r,s})| \geq \frac{|V(D_{r,s})|}{2} + 1$.

**Case (ii):** Suppose r < s.

Then $\{\{u, v, v_1, v_2, \ldots, v_{\frac{r+s+2}{2}}\}, \{u_1\}, \{u_2\}, \ldots, \{u_r\}, \{v_{\frac{r+s+2}{2}+1}\}, \ldots, \{v_s\}\} = \{V_1, V_2, \ldots, V_k\}$, where $k = 1 + \left(s - \frac{r+s-2}{2}\right) + r = \frac{r+s+2}{2} + 1$ is an ET-partition with $T_{V_i} = 1$, for all i, $1 \leq i \leq k$. Therefore, $ET(D_{r,s}) \geq \frac{|V(D_{r,s})|}{2} + 1$.

Suppose ET $(D_{r,s}) > \frac{|V(D_{r,s})|}{2} + 1$. Let ET $(D_{r,s})$ = t ( say). Let $\pi = \{V_1, V_2, \ldots, V_t\}$ be a maximum equi-toughness partition. If $|V_i| \geq 2$, for all i, then $|V(D_{r,s})| \geq 2t > 2\left(\frac{|V(D_{r,s})|}{2} + 1\right)$, a contradiction. Suppose there are (t – 3) sets each of which has cardinality at least two and the other three sets in π are of cardinality 1. Then

$$|V(D_{r,s})| \geq 2(t-3)+3 = 2t -3 \geq 2\left(\frac{|V(D_{r,s})|}{2}+2\right)-3 = \frac{|V(D_{r,s})|}{2}+1,$$ a contradiction. Therefore, the number of sets in $\pi$ each of which has cardinality at least two is at most t - 4. Thus, there are at least 4 sets in $\pi$ which have cardinality 1. Therefore there are at least four pendent vertices which appear as singleton sets in $\pi$. Therefore $T_{V_i} = 1$, for all i, $1 \leq i \leq k$.

**Subcase (1):** Let $u,v \in V_i$, for some i.

Then $\omega(V - V_i) = |V - V_i|$, since $T_{V_i} = 1$, for all i. Therefore $|V_i| = |V - V_i|$. Suppose $V_i$ has t pendent vertices. Then t + 2 = r + s + 2 - (t +2). Thus, $t = \frac{r+s+2}{2}$. Therefore ET $(D_{r,s})$ $= 1 + (r + s - t) = 1 + \frac{r+s+2}{2}$.

**Subcase (2):** Let $u \in V_i$ and $v \in V_j$. Let $V_i$ contains $t_1$ pendent vertices adjacent to u and $t_2$ pendent vertices adjacent to v. Then $\omega(V - V_i) = (r - t_1) + 1$.

$|V_i| = t_1 + t_2 + 1$. Therefore, $r - t_1 + 1 = t_1 + t_2 + 1$.

Thus, $r = t_2 + 2t_1$. Let $V_j$ contain $t_3$ pendent vertices adjacent to u and $t_4$ pendent vertices adjacent to v. Then $\omega(V - V_j) = (s - t_4) + 1$. $|V_j| = t_3 + t_4 + 1$. Therefore, $s - t_4 + 1 = t_3 + t_4 + 1$.

Thus, $s = t_3 + 2t_4$.

Therefore, $|\pi| = 2 + (r + s) - (t_1 + t_2 + t_3 + t_4) = 2 + t_1 + t_4$.

Consider the following LPP,

Maximize $t_1 + t_4$

Subject to the constraints

$2t_1 + t_2 = r$

$2t_4 + t_3 = s$. The optimal solution of this problem is $t_1 = \frac{r}{2}$, $t_4 = \frac{s}{2}$ and $\max|\pi| = 2 + \frac{r+s}{2} = \frac{|V(D_{r,s})|}{2} + 1$.

**Case (B):** r, s are odd

**Case (i):** Suppose r = s.

Then $\{\{u, v, v_1, v_2, ..., v_{s-1}\}, \{u_1\}, \{u_2\}, ..., \{u_r\}, \{v_s\}\} = \{V_1, V_2, ..., V_k\}$, where $k = s + 2 = \frac{r+s}{2} + 2 = \frac{r+s+2}{2} + 1$ is an ET-partition with $T_{V_i} = 1$, for all i, $1 \le i \le k$. Therefore, $ET(D_{r,s})| \ge \frac{|V(D_{r,s})|}{2} + 1$.

**Subcase (1):** Let $u, v \in V_i$, for some i. Arguing as in the subcase (1) of case (A), we have $ET(D_{r,s}) = 1 + \frac{r+s+2}{2}$.

**Subcase (2):** Let $u \in V_i$ and $v \in V_j$. Arguing as in the subcase (2) of case (A), we get, $|\pi| = 2 + (r + s) - (t_1 + t_2 + t_3 + t_4) = 2 + t_1 + t_4$, where $2t_1 + t_2 = r$, $2t_4 + t_3 = r$. Since r is odd, $t_2 \ge 1$ and $t_3 \ge 1$. Therefore, $t_1 + t_4 < r = \frac{r+s}{2}$.

Therefore $|\pi| = 2 + t_1 + t_4 < 2 + r = \frac{|V(D_{r,s})|}{2} + 1$. Therefore, if u and v belong to different elements of $\pi$, then $|\pi|$ will not be maximum.

**Case (ii):** Suppose r < s. Then $\{\{u, v, v_1, v_2, ..., v_{\frac{r+s+2}{2}}\}, \{u_1\}, \{u_2\}, ..., \{u_r\}, \{v_{\frac{r+s+2}{2}+1}\}, ..., \{v_s\}\} = \{V_1, V_2, ..., V_k\}$, where $k = 1 + \left(s - \frac{r+s-2}{2}\right) + r = \frac{r+s+2}{2} + 1$ is an

ET-partition with $T_{V_i} = 1$, for all i, $1 \le i \le k$. Therefore, $ET(D_{r,s}) \ge \frac{|V(D_{r,s})|}{2} + 1$.

**Subcase (1)**: Let u, v ∈ $V_i$, for some i. Arguing as in the subcase (1) of case (A), we have ET $(D_{r,s}) = 1 + \frac{r+s+2}{2}$.

**Subcase (2)**: Suppose u∈ $V_i$ and v ∈ $V_j$. Arguing as in subcase (2) of case (i) of case (B), we get $|\pi|$ is not maximum.

**Case(C):** r and s are of opposite parity.

Without loss of generality, let r be odd and s be even. Suppose $|\pi| \ge \left\lceil \frac{|V(D_{r,s})|}{2} \right\rceil = \frac{|V(D_{r,s})|+1}{2}$. Suppose ET $(D_{r,s}) > \frac{|V(D_{r,s})|+1}{2}$.

Therefore, ET $(D_{r,s}) \ge \frac{|V(D_{r,s})|+3}{2}$. Suppose there are (t - 2) elements of π having cardinality at least 2.

Then $|\pi| = |V_1| + |V_2| + \ldots + |V_t| \ge 2\,(t-2) + 2 = 2\left(\frac{|V(D_{r,s})|+3}{2}\right) - 2 + 2 = |V(D_{r,s})| + 1$, a contradiction.

Therefore π has at least 3 singletons. Therefore $T_{V_i} = 1$ for all i.

**Subcase (i):** Suppose u,v ∈ $V_i$, for some i. But $\omega(V - V_i) = |V - V_i|$. Let $V_i$ have $\ell$ pendent vertices. Then $\ell + 2 = r + s -$

1. Therefore, $2\ell = r + s - 2$, a contradiction (since left hand side is even and right hand side is odd). Therefore, u and v can not belong to the same element of $\pi$.

**Subcase (2):** Suppose $u \in V_i$ and $v \in V_j$. Let $V_i$ contains $t_1$ pendent vertices adjacent to u and $t_2$ pendent vertices adjacent to v. Then $\omega(V - V_i) = (r - t_1) + 1$. $|V_i| = t_1 + t_2 + 1$. Therefore, $r - t_1 + 1 = t_1 + t_2 + 1$. Thus, $r = t_2 + 2t_1$. Let $V_j$ contain $t_3$ pendent vertices adjacent to u and $t_4$ pendent vertices adjacent to v. Then $\omega(V - V_i) = (s - t_4) + 1$. $|V_j| = t_3 + t_4 + 1$. Therefore, $s - t_4 + 1 = t_3 + t_4 + 1$. Thus, $s = t_3 + 2t_4$. Therefore, $|\pi| = 2 + (r + s) - (t_1 + t_2 + t_3 + t_4) = 2 + t_1 + t_4$.

Consider the following IPP,
Maximize $t_1 + t_4$
Subject to the constraint
$2t_1 + t_2 = r$
$2t_4 + t_3 = s$,
Since r is odd, $t_2 \geq 1$ and s is even, $t_3$ is even. The solution for the above IPP is $t_1 = \frac{r-1}{2}$ and $t_4 = \frac{s}{2}$. Thus, $|\pi| = \frac{r+s-1}{2} + 2 = \frac{r+s+3}{2}$. Hence $ET(D_{r,s}) = \frac{|V(D_{r,s})|+1}{2}$.

**Definition 2.13:** Let G be graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$. The Mycielski transformation of G, denoted $\mu(G)$, has for its vertex set, the set $\{x_1, x_2 \ldots, x_n, y_1, y_2, \ldots, y_n, z\}$. As for adjacency, $x_i$ is adjacent with $x_j$ in $\mu(G)$ if and only if $v_i$ is adjacent with $v_j$ in G, $x_i$ is adjacent with $y_j$ in $\mu(G)$ if and only if $v_i$ is adjacent with $v_j$ in G, and $y_i$ is adjacent with z in $\mu(G)$ for all $i \in \{1, 2, \ldots, n\}$.

**Corollary 2.14:** If G is any connected graph of order n, then ET ($\mu$ (G)) = |V ($\mu$(G))|, since $\kappa(\mu$ (G)) $\geq 2$.

**Observations 2.15:**

(i). If G is hamiltonian, then ET (G) = n, since every hamiltonian graph is 2-connected.

(ii). If G is k-regular of order 2k +1, then ET (G) = n, since G is hamiltonian.

(iii). Let G be connected. Then, ET (G) $\leq$ ET ($G^n$)($n \geq 2$), since $G^n$ is a k ($\geq$2)-connected graph.

(iv). If G is a connected graph with $\kappa$(G) $\geq 2$, then ET (G) $\leq$ ET (L (G)), where L (G) denotes the line graph of G.

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