Fuzzy Inner Product Spaces

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Abstract

In this paper fuzzy inner product on a real vector space is introduced. The notion of fuzzy inner product is defined. Some of its properties are studied.

Keywords: Fuzzy norm, Fuzzy normed linear space, Fuzzy inner product space.

1 Introduction

The concept of metric space is based on the distance between two points. Menger [8] defined statistical metric space based on the concept that the probality of the distance between x and y is less than t. Schweizer and Sklar [9] introduced the concept of t-norm. They generalized statistical metric space and defined probablistic metric space. Kromosil and Michalak [7] generalized the concept of probablistic metric space which is called a KM fuzzy metric space. George and Veeramani [6] modified KM fuzzy metric space. Also George defined fuzzy normed space. Modified definition of fuzzy normed is given in [1]. In this paper the concept of fuzzy inner product is space is defined. Also the fuzzy normed linear space induced by a fuzzy inner product is studied.

2 Preliminary Results

Definition 1 *A binary operation* $* : [0,1] \times [0,1] \longrightarrow [0,1]$ *is a t-norm if* $*$ *satisfies the following conditions:*

1. $*$ *is associative and commutative*
2. $a * 1 = a$ *for all* $a \in [0,1]$

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3. $a * b \leq c * d$ *whenever* $a \leq c$ *and* $b \leq d$ *for all* $a, b, c, d \in [0,1]$

Definition 2 *A 3-tuple* $(X, M, *)$ *is said to be a fuzzy metric space if* X *is any arbitrary set,* $*$ *is continuous t-norm and* M *is a fuzzy set on* $X^2 \times (0, \infty)$ *satisfying the following conditions:*

1. $M(x, y, t) > 0$ *for all* $x, y \in X$ *and* $t \in (0, \infty)$.
2. $M(x, y, t) = 1$ *if and only if* $x = y$.
3. $M(x, y, t) = M(y, x, t)$.
4. $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$.
5. $M(x, y, \cdot) : (0, \infty) \longrightarrow [0, 1]$ *is continuous*

for all $x, y, z \in X$ *and* $t, s > 0$.

$M(x, y, \cdot)$ is non dcreasing for all $x, y, \in X$. Let $(X, M, *)$ be a fuzzy metric space. An open ball $B(x, r, t)$ with centre $x \in X$ and radius $r, 0 < r < 1, t > 0$, is defined as $B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}$. Let $\tau = \{A \subset X : x \in A$ if and only if there exist there exist $r, t > 0, 0 < r < 1$, such that $B(x, r, t) \subset A\}$. Then τ is a topology on X .In a fuzzy metric space every open ball is an open set. The topology τ is first countable and and also Hausdorff. A sequence (x_n) in a fuzzy metric space $(X, M, *)$ converges to $x \in X$ if for given $r, t > 0, 0 < r < 1$, there exists a positive integer n_0 such that $M(x_n, x, t) > 1 - r, n \geq n_0$. Clearly a sequence (x_n) in a fuzzy metric space is convergent to $x \in X$ if and only if $M(x_n, x, t) \to 1$ as $n \to \infty$. A sequence (x_n) in a fuzzy metric space $(X, M, *)$ is said to be a Cauchy sequence if for given $r, t > 0, 0 < r < 1$, there exists a positive integer n_0 such that $M(x_n, x_m, t) > 1 - r$ for all $m, n \geq n_0$. A fuzzy metric space is complete if every Cauchy sequence in it converges.

Definition 3 *A 3-tuple* $(X, N, *)$ *is said to be a fuzzy normed linear space if* X *is real or complex linear space,* $*$ *is continuous t-norm and* N *is a fuzzy set on* $X \times (0, \infty)$ *satisfying the following conditions:*

1. $N(x, t) > 0$ *for all* $x \in X$ *and* $t \in (0, \infty)$.
2. $N(x, t) = 1$ *if and only if* $x = 0$.

3. $N(kx,t) = M\left(x, \frac{t}{|k|}\right)$.

4. $N(x+y, t+s) \geq N(x,t) * N(y,s)$.

5. $N(x,\cdot) : (0,\infty) \longrightarrow [0,1]$ *is continuous*

for all $x, y, \in X$, $t, s > 0$ and k is a scalar.

We write a fuzzy normed linear space briefly as F-normed space. Let $(X, N, *)$ be an F-normed space. For any $t > 0$ and for all $x, y \in X$ define $M(x,y,t) = N(x-y,t)$. Then $(X, M, *)$ is a fuzzy metric space. A sequence (x_n) in an F-normed space $(X, N, *)$ converges to $x \in X$ if for given $r, t > 0, 0 < r < 1$, there exists a positive integer n_0 such that $N(x_n - x, t) > 1 - r, n \geq n_0$. The sequence (x_n) in an F-normed space is convergent to $x \in X$ if and only if $N(x_n - x, t) \to 1$ as $n \to \infty$. In an F-normed space, the operation of addition is jointly continuous. A sequence (x_n) in an F-normed space $(X, N, *)$ is said to be a F-Cauchy sequence if for given $r, t > 0, 0 < r < 1$ there exists a positive integer n_0 such that $N(x_n - x_m, t) > 1 - r$ for all $m, n \geq n_0$. An F-normed space is said to be complete if every F-Cauchy sequence in X converges to an element in X. A complete F-normed space is called as F-Banach space. A linear transformation T from an F-normed space $(X, N, *)$ to an F-normed space $(X', N', *)$ is said to be bounded if there exists $k > 0$, such that $N'(T(x), kt) \geq N(x,t)$ for all $x \in X$ and $t > 0$.

3 Fuzzy Inner Product Spaces

Definition 4 *A triplet* $(X, J, *)$ *is said to be a fuzzy inner product space if* X *is a real linear space,* $*$ *is continuous t-norm and* J *is a fuzzy set on* $X^2 \times (0,\infty)$ *satisfying the following conditions:*

1. $J(x,y,t) > 0$ *for all* $x, y \in X$ *and* $t \in (0,\infty)$.

2. $J(x,x,t) = 1$ *if and only if* $x = 0$.

3. $J(x,y,t) = J(y,x,t)$.

4. $J(\alpha x, \beta y, t) \geq J\left(x, y, \frac{t}{|\alpha\beta|}\right)$, α, β are scalars.

5. $J(x+y, z, t+s) \geq J(x,z,t) * M(y,z,s)$

6. $J\left(x, y, \sqrt{st}\right) \geq M(x,x,t) * M(y,y,s)$

7. $J(x,y,\cdot):(0,\infty)\longrightarrow[0,1]$ *is continuous for all* $x,y,z\in X$ *and* $t,s>0$

Example 1. Consider R^2, let $x,y\in R^2$, where $x=(x_1,x_2)$ and $y=(y_1,y_2)$. Define $J(x,y,t)=\left(\exp\left(\frac{|x_1y_1+x_2y_2|}{t}\right)\right)^{-1}$. Then $(R^2,J,\min)$ is a fuzzy inner product space.
Clearly $J(x,y,t)>0$ and $J(x,y,t)=J(y,x,t)$. Also $J(x,x,t)=1\Leftrightarrow x=0$.
Now

$$\begin{aligned}&J(\alpha x,\beta y,t)\\ =&\left(\exp\left(\frac{|\alpha\beta x_1y_1+\alpha\beta x_2y_2|}{t}\right)\right)^{-1}\\ =&\left(\exp\left(\frac{|\alpha\beta||x_1y_1+x_2y_2|}{t}\right)\right)^{-1}\\ =&\left(\exp\left(\frac{|x_1y_1+x_2y_2|}{\frac{t}{|\alpha\beta|}}\right)\right)^{-1}=J\left(x,y,\frac{t}{|\alpha\beta|}\right)\end{aligned}$$

Without loss of generality assume that $J(x,z,t)\leq J(y,z,s)$
Then $\left(\exp\left(\frac{|x_1z_1+x_2z_2|}{t}\right)\right)^{-1}\leq\left(\exp\left(\frac{|y_1z_1+y_2z_2|}{s}\right)\right)^{-1}$,
i.e., $\frac{|x_1z_1+x_2z_2|}{t}\geq\frac{|y_1z_1+y_2z_2|}{s}$, i.e., $\frac{|x_1z_1+x_2z_2|}{t}\geq\frac{|(x_1+y_1)z_1+(x_2+y_2)z_2|}{s+t}$,
i.e., $\left(\exp\left(\frac{|x_1z_1+x_2z_2|}{t}\right)\right)^{-1}\leq\left(\exp\left(\frac{|(x_1+y_1)z_1+(x_2+y_2)z_2|}{s+t}\right)\right)^{-1}$,
i.e., $J(x,z,t)\leq J(x+y,z,t+s)$. Therefore $\min\{J(x,z,t),J(y,z,s)\}\leq J(x+y,z,t+s)$
To prove axiom 6, without loss of generality assume that $J(x,x,t)\leq J(y,y,s)$.
Then $\left(\exp\left(\frac{|x_1^2+x_2^2|}{t}\right)\right)^{-1}\leq\left(\exp\left(\frac{|y_1^2+y_2^2|}{s}\right)\right)^{-1}$,
i.e., $\frac{|x_1^2+x_2^2|}{t}\geq\frac{|y_1^2+y_2^2|}{s}$, i.e., $\frac{st|x_1^2+x_2^2|}{t^2}\geq|y_1^2+y_2^2|$
i.e., $\frac{|x_1^2+x_2^2|^2}{t^2}\geq\frac{|x_1^2+x_2^2||y_1^2+y_2^2|}{st}\geq\frac{|x_1y_1+x_2y_2|^2}{st}$
since $|x_1^2+x_2^2|\,|y_1^2+y_2^2|\geq|x_1y_1+x_2y_2|^2$
This implies $\left(\exp\left(\frac{|x_1^2+x_2^2|}{t}\right)\right)^{-1}\leq\left(\exp\left(\frac{|x_1y_1+x_2y_2|}{\sqrt{st}}\right)\right)^{-1}$,

i.e.,$J(x,x,t) \leq J\left(x,y,\sqrt{st}\right)$
Therefore $\min\{J(x,x,t), J(y,y,s)\} \leq J\left(x,y,\sqrt{st}\right)$.
Clearly $J(x,y,\cdot) : (0,\infty) \longrightarrow [0,1]$ is continuous. Therefore $\left(R^2, J, \min\right)$ is a fuzzy inner product space.

It canbe easily verified that above example holds good with the t-norm $a * b = ab$.

Example 2. Let $X = R$ and let $x, y \in R$. Define for $t > 0, J(x,y,t) = 1$ if $x = 0$ or $y = 0$, $J(x,y,t) = l$ if $x \neq 0$ and $y \neq 0, 0 < l < 1$. Then $(R, J, \min)$ is a fuzzy inner product space.

Axioms 1, 2 and 3 can be easily proved. If $x \neq 0$ and $y \neq 0$, then for $t > 0, J(\alpha x, \beta y, t) = l$ and $J\left(x,y,\frac{t}{|\alpha\beta|}\right) = l$. Therefore $J(\alpha x, \beta y, t) = J\left(x,y,\frac{t}{|\alpha\beta|}\right)$. If $x = 0$ and $y \neq 0$ then for $t > 0$, $J(\alpha x, \beta y, t) = 1$ and $J\left(x,y,\frac{t}{|\alpha\beta|}\right) = 1$. Therefore $J(\alpha x, \beta y, t) = J\left(x,y,\frac{t}{|\alpha\beta|}\right)$. Similarly we can prove for the case $x = 0$ or $y = 0$. Suppose $x \neq 0$, $y \neq 0$ and $z \neq 0$. Let $t, s > 0$. If $x + y \neq 0$ then $J(x+y, z, t+s) = l, J(x,z,t) = l$ and $J(y,z,t) = l$.
Therefore $J(x+y,z,t+s) = \min\{J(x,z,t), M(y,z,s)\}$.
If $x + y = 0$ then $J(x+y,z,t+s) = 1, J(x,z,t) = l$ and $J(y,z,t) = l$. Therefore $J(x+y,z,t+s) > \min\{J(x,z,t), M(y,z,s)\}$. $J(x,y,t) = J(y,x,t)$. Suppose $x = 0$, $y \neq 0$ and $z \neq 0$. Let $t, s > 0$, then $J(x+y,z,t+s) = l$, $J(x,z,t) = 1$ and $J(y,z,t) = l$.
Therefore $J(x+y,z,t+s) = \min\{J(x,z,t), M(y,z,s)\}$. In the sameway other cases can be proved. Now If $x \neq 0$, $y \neq 0$ and $t, s > 0$. Then $J\left(x,y,\sqrt{st}\right) = l, J(x,x,t) = l$ and $J(y,y,t) = l$
Therefore $J\left(x,y,\sqrt{st}\right) = \min\{J(x,x,t), M(y,y,s)\}$. Suppose $x = 0$, $y \neq 0$ then
$J\left(x,y,\sqrt{st}\right) = 1, J(x,x,t) = 1$ and $J(y,y,t) = l$.
This implies $J\left(x,y,\sqrt{st}\right) > \min\{J(x,x,t), M(y,y,s)\}$. Similarly other cases can be proved.
Clearly $J(x,y,\cdot) : (0,\infty) \longrightarrow [0,1]$ is continuous. Therefore $(R, J, \min)$ is a fuzzy inner product space.

Example 3. Let $(X, \langle\rangle)$ be an inner product space. Define $J(x,y,t) = \frac{t}{t+|\langle x,y\rangle|}$. Then $(X, J, \min)$ is a fuzzy inner product space.
Clearly $J(x,y,t) > 0$. Also $J(x,x,t) = 1 \Leftrightarrow x = 0$. and $J(x,y,t) = J(y,x,t)$.
Now $J(\alpha x, \beta y, t) = \frac{t}{t+|\langle \alpha x, \beta y\rangle|} = \frac{t}{t+|\alpha\beta||\langle \alpha x, \beta y\rangle|} = \frac{\frac{t}{|\alpha\beta|}}{\frac{t}{|\alpha\beta|}+|\langle x,y\rangle|}$

$= J\left(x, y, \frac{t}{|\alpha\beta|}\right)$. To prove axiom 5, with out loss of generality assume that $J(x,z,t) \leq J(y,z,s)$.
Then $\frac{t}{t+|\langle x,z\rangle|} \leq \frac{s}{s+|\langle y,z\rangle|}$, i.e., $\frac{s|\langle x,z\rangle|}{t} \geq |\langle y,z\rangle|$
i.e., $\left(1+\frac{s}{t}\right)|\langle x,z\rangle| \geq |\langle x,z\rangle| + |\langle y,z\rangle| \geq |\langle x+y,z\rangle|$,
i.e., $1 + \frac{|\langle x,z\rangle|}{t} \geq 1 + \frac{|\langle x+y,z\rangle|}{t}$, i.e., $\frac{t}{t+|\langle x,z\rangle|} \leq \frac{s+t}{(s+t)+|\langle x+y,z\rangle|}$, i.e., $J(x,z,t) \leq J(x+y,z,t+s)$.
Therefore $\min\{J(x,z,t), J(y,z,s)\} \leq J(x+y,z,t+s)$.
Now with out loss of generality assume that $J(x,x,t) \leq J(y,y,s)$.
Then $\frac{t}{t+|\langle x,x\rangle|} \leq \frac{s}{s+|\langle y,y\rangle|}$, i.e., $\frac{s|\langle x,x\rangle||\langle x,x\rangle|}{t} \geq |\langle x,x\rangle||\langle y,y\rangle|$.
i.e., $\frac{st|\langle x,x\rangle|^2}{t^2} \geq |\langle x,y\rangle|^2$ since $|\langle x,y\rangle|^2 \leq |\langle x,x\rangle||\langle y,y\rangle|$
This implies $1 + \frac{|\langle x,x\rangle|}{t} \geq 1 + \frac{|\langle x,y\rangle|}{\sqrt{st}}$. i.e., $\frac{t}{t+|\langle x,x\rangle|} \leq \frac{\sqrt{st}}{\sqrt{st}+|\langle x,y\rangle|}$.
i.e., $J(x,x,t) \leq J\left(x,y,\sqrt{st}\right)$.
Therefore $\min\{J(x,x,t), J(y,y,s)\} \leq J\left(x,y,\sqrt{st}\right)$.
Clearly $J(x,y,\cdot) : (0,\infty) \longrightarrow [0,1]$
is continuous. Therefore $(X, J, \min)$ is a fuzzy inner product space.

With the help of the fuzzy inner product on a linear space X, we can define a F-norm as follows.

Theorem 1. Let $(X, J, \min)$ is a fuzzy inner product space. Define $N(x,t) = J\left(x,x,t^2\right)$. Then $(X, N, \min)$ is an F-normed space.
Clearly $N(x,t) = J\left(x,x,t^2\right) > 0$. Also $N(x,t) = 1 \Leftrightarrow J\left(x,x,t^2\right) = 1 \Leftrightarrow x = 0$.
Now $N(\alpha x,t) = J\left(\alpha x, \alpha x, t^2\right) = J\left(x,x,\frac{t^2}{|\alpha|^2}\right) = N\left(x, \frac{t}{|\alpha|}\right)$.
It is clear that $N(x+y,t+s) = J\left(x+y,x+y,(t+s)^2\right)$
$= J\left(x+y,x+y,t^2+s^2+2ts\right)$
$\geq \min\{J\left(x,x+y,t^2+ts\right), J\left(y,x+y,s^2+ts\right)\}$.
$\geq \min\{J\left(x,x,t^2\right), J(y,x,ts), J\left(y,y,s^2\right), J(y,x,ts)\}$
$= \min\{J\left(x,x,t^2\right), J(y,x,ts), J\left(y,y,s^2\right),\}$.
$\geq \min\{J\left(x,x,t^2\right), J\left(x,x,t^2\right), J\left(y,y,s^2\right), J\left(y,y,,s^2\right)\}$
$= \min\{J\left(x,x,t^2\right), J\left(y,y,s^2\right)\} = \min\{N(x,t), N(y,s)\}$.
Clearly $N(x,\cdot) : (0,\infty) \longrightarrow [0,1]$ is continuous. Therefore $(X, N, \min)$ is an F-normed space. □

Proposition. Let $(X, J, \min)$ is a fuzzy inner product space. Then $x, y \in X$, $t > 0$, the following are true

i) $J(0,y,t) \geq J\left(x,y,\frac{t}{2}\right)$
ii) $J(0,0,t) \geq J\left(x,y,\frac{t}{4}\right)$.
Proof: $J(0,y,t) = J(x-x,y,t) \geq \min\{J\left(x,x,\frac{t}{2}\right), J\left(-x,y,\frac{t}{2}\right)\} = J\left(x,x,\frac{t}{2}\right)$. Also $J(0,0,t) = J(x-x,0,t) \geq \min\{J\left(x,0,\frac{t}{2}\right), J\left(-x,0,\frac{t}{2}\right)\} = J\left(x,0,\frac{t}{2}\right) \geq J\left(x,y,\frac{t}{4}\right)$

Proposition. Let $(X,J,\min)$ is a fuzzy inner product space. Then $x,y \in X, t>0$, $\min\{J\left(x+y,x+y,t^2\right), J\left(x-y,x-y,t^2\right)\} \geq \min\{J\left(x,x,\frac{t^2}{4}\right), J\left(y,y,\frac{t^2}{4}\right)\}$.

Since we are able to define a norm on X with the help of the inner product , the fuzzy inner product space $(X,J,\min)$ becomes a F- normed space $(X,N,\min)$. If the fuzzy inner product space is complete in this F-norm then X is called a Fuzzy Hilbert space.

Example 4. In example (1) we proved $\left(R^2,J,\min\right)$ is a fuzzy inner product space with the inner product $J(x,y,t) = \left(\exp\left(\frac{|x_1y_1+x_2y_2|}{t}\right)\right)^{-1}$ where $x=(x_1,x_2)$ and $y=(y_1,y_2)$. Now we prove that it is a fuzzy Hilbert space with the induced F-norm $N(x,t) = J\left(x,x,t^2\right)$.

We have to show that any Cauchy sequence in $\left(R^2,N,\min\right)$ converges in this norm. Let $\left(x^{(n)}\right)$ be a Cauchy sequence in $\left(R^2,N,\min\right)$. Then any term $x^{(n)}$ of this sequence is of the form $x^{(n)} = \left(\xi_1^{(n)},\xi_2^{(n)}\right)$. Since $\left(x^{(n)}\right)$ is a Cauchy sequence we have $N\left(x^{(n)}-x^{(m)}x,t\right) \to 1$ as $n\to\infty$. This implies $\left(\exp\left(\frac{(\xi_1^n-\xi_1^m)^2+(\xi_2^n-\xi_2^m)^2}{t^2}\right)\right)^{-1} \to 1$as $n\to\infty$. i.e.,$\frac{(\xi_1^n-\xi_1^m)^2+(\xi_2^n-\xi_2^m)^2}{t^2} \to 0$ as $n\to\infty$. This implies $\xi_i^n - \xi_i^m \to 0, i=1,2$. This showes that for $i=1,2$, $\left(\xi_i^{(n)}\right)$ is a Cauchy sequence in R. Since R is complete implies $\left(\xi_i^{(n)}\right), i=1,2$ converges in R.Let $\xi_i^{(n)} \to \xi_i$ as $n\to\infty$. Define $x = (\xi_1,\xi_2) \in R^2$. Now $N\left(x^{(n)}-x,t\right) = J\left(x^{(n)}-x,x^{(n)}-x,t^2\right) = \left(\exp\left(\frac{(\xi_1^n-\xi_1)^2+(\xi_2^n-\xi_2)^2}{t^2}\right)\right)^{-1} \to 1$ as $n\to\infty$. This implies the sequence $\left(x^{(n)}\right)$ converges to x. Therfore $\left(R^2,J,\min\right)$ is a fuzzy Hilbert space.

Now we define fuzzy inner product in product spaces.

Theorem 2. Let $(X,J_1,*)$ and $(Y,J_2,*)$ be fuzzy inner product spaces. Then $(X\times Y,J,*)$ is a fuzzy inner product space where

$J((x_1,y_1),(x_2,y_2),t) = J_1(x_1,x_2,t) * J_2(y_1,y_2,t)$, $x_1,x_2 \in X$ and $y_1,y_2 \in Y$. Also $X \times Y$ is a Hilbert space if X and Y are Hilbert spaces.

Proof: Clearly $J((x_1,y_1),(x_2,y_2),t) > 0$. Also

$$J((x_1,y_1),(x_1,y_1),t) = 1 \Leftrightarrow J_1(x_1,x_1,t) = J_2(y_1,y_1,t) = 1 \Leftrightarrow (x_1,y_1) = 1$$

Clearly $J((x_1,y_1),(x_2,y_2),t) = J((x_2,y_2),(x_1,y_1)t)$.
Now. $J(\alpha(x_1,y_1),\beta(x_2,y_2),t)$
$= J_1(\alpha x_1,\beta x_2,t) * J_2(\alpha y_1,\beta y_2,t) = J_1\left(x_1,x_2,\frac{t}{|\alpha\beta|}\right) * J_2\left(y_1,y_2,\frac{t}{|\alpha\beta|}\right)$
$= J\left((x_1,y_1),(x_2,y_2),\frac{t}{|\alpha\beta|}\right)$. It is clear that

$$\begin{aligned}
& J((x_1,y_1)+(x_2,y_2),(x_3,y_3),t+s) \\
= \; & J((x_1+x_2,y_1+y_2),(x_3,y_3),t+s) \\
= \; & J_1(x_1+x_2,x_3,t+s) * J_2(y_1+y_2,y_3,t+s) \\
\geq \; & J_1(x_1,x_3,t) * J_1(x_2,x_3,s) * J_2(y_1,y_3,t) * J_2(y_2,y_3,s) \\
= \; & J((x_1,y_1),(x_3,y_3),t) * J((x_2,y_2),(x_3,y_3),t)
\end{aligned}$$

Now

$$\begin{aligned}
& J\left((x_1,y_1),(x_2,y_2),\sqrt{st}\right) \\
= \; & J_1\left(x_1,x_2,\sqrt{st}\right) * J_2\left(y_1,y_2,\sqrt{st}\right) \\
\geq \; & J_1(x_1,x_1,t) * J_1(x_2,x_2,s) * J_2(y_1,y_1,t) * J_2(y_2,y_2,s) \\
= \; & J((x_1,y_1),(x_1,y_1),t) * J_2((x_2,y_2),(x_2,y_2),s)
\end{aligned}$$

Clearly $J((x_1,y_1),(x_2,y_2),\cdot) : (0,\infty) \longrightarrow [0,1]$ is continuous. Therefore $(X \times Y, J, *)$ is a fuzzy inner product space. Let $N((x,y),t) = J((x,y),(x,y),t)$, $N_1(x,t) = J_1(x,x,t))$ and $N_2(x,t) = J_2(x,x,t))$ be the induced fuzzy norms. Then $N((x,y),t) = N_1(x,t) * N_2(y,t)$ .Suppose X and Y are Fuzzy Hilbert spaces. Let (x_n,y_n) be an F-Cauchy sequence in $X \times Y$. Then for given $r,t > 0, 0 < r < 1$ there exists a positive integer k such that $N((x_n,y_n)-(x_m,y_m),t) > 1-r, m,n, \geq k$. This implies $N_1(x_n-x_m,t) > 1-r$ and $N_2(y_n-y_m,t) > 1-r$ for all $m,n, \geq k$.Since X and Y Fuzzy Hilbert spaces , $x_n \to x$ and $y_n \to y$ for some $x \in X$ and $y \in Y$ .Therefore $N((x_n,y_n)-(x,y),t) \to 1$ as $n \to \infty$. Hence (x_n,y_n) converges to (x,y). Therefore $X \times Y$ is a Fuzzy Hilbert space. □

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