New Constructions of Super Edge Bimagic Labeling

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Abstract

A graph $G(p,q)$ is said to be total edge bimagic with two common edge count k_1 and k_2 if there exists a bijection $f : V(G) \cup E(G) \to \{1,2,\ldots,p+q\}$ such that for each edge $uv \in E(G)$, $f(u)+f(v)+f(uv) = k_1$ or k_2. A total edge-bimagic graph is called super edge-bimagic if $f(V(G)) = \{1,2,\ldots,p\}$. In this paper we define new types of super edge bimagic labeling and prove some interesting results related to super edge bimagic labeling. Also its relationship with cordial labeling is studied.

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1 Introduction

A labeling of a graph G is an assignment f of labels to either the vertices or the edges or both subject to certain conditions. Labeled graphs are becoming an increasingly useful family of Mathematical Models from a broad range of applications. Graph labeling was first introduced in the late 1960's. They are useful in many Coding Theory problems, including the design of good Radar Type Codes, Missile Guidance Codes and Convolution Codes with optimal autocorrelation properties. They facilitate the optimal nonstandard encoding of integers. An enormous body of literature has grown around the subject in the last thirty years. They gave birth to families of graphs with attractive names such as graceful, harmonious, felicitous, elegant, cordial, magic, antimagic and prime labeling. A useful survey to know about the numerous graph labeling methods is the one by J.A. Gallian recently (2009) [7].

All graphs considered here are finite, simple and undirected. A (p, q) graph $G = (V, E)$ with p vertices and q edges is called total edge magic if there is a bijection function $f : V \cup E \to \{1, 2, \ldots, p+q\}$ such that for any edge uv in E we have a constant k with $f(u) + f(uv) + f(v) = k$. A total edge-magic graph is called super edge-magic if $f(V(G)) = \{1, 2, \ldots, p\}$. Wallis [10] called super edge-magic as strongly edge-magic. Edge bimagic totally labeling was introduced by J. Baskar Babujee [1] and studied in [2] as $(1, 1)$ edge bimagic labeling. A graph $G(p, q)$ with p vertices and q edges is called total edge bimagic if there exists a bijection $f : V \cup E \to \{1, 2, \ldots, p+q\}$ such that for any edge $uv \in E$, we have two constants k_1 and k_2 with $f(u) + f(v) + f(uv) = k_1$ or k_2. A total edge-bimagic graph is called super edge-bimagic if $f(V(G)) = \{1, 2, \ldots, p\}$. Super edge-bimagic labeling for the trees path, star-$K_{1,n}$, $K_{1,n,n}$ are proved in [3].

Example 1.1. *Super edge bimagic labeling of $P_7$is given in Figure 1.*

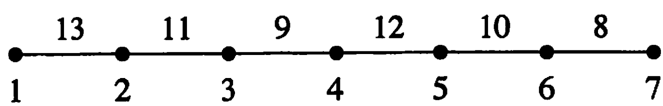

Figure 1: $k_1 = 16, k_2 = 21$

Super edge-bimagic labeling for cycles, Wheel graph, Fan graph, Gear graph, Maximal Planar class-$Pl_n : n = 5$, $K_{1,m} \cup K_{1,n}$ $(m, n \geq 1)$, $P_n \cup P_{n+1}$ $(n \geq 2)$, $C_3 \cup K_{1,n}$ $(n \geq 1)$, $P_m \odot K_{1,n}$, $(3, n)$-kite graph $(n \geq 2)$, $P_n + N_2 (n \geq 3)$, $P_2 \cup mK_1 + N_2$ $(m \geq 1)$ are proved in [4, 5, 6].Let f be a function from the vertices of G to $\{0, 1\}$ and for each edge xy assign the label $|f(x) - f(y)|$. Call f a cordial labeling of G if the number of vertices labeled 0 and the number of vertices labeled 1 differ by at most 1, and the number of edges labeled 0 and the number of edges labeled 1 differ at most by 1. In this paper we define new types of edge bimagic total labeling and prove some interesting results related to super edge bimagic labeling. Also its relationship with cordiallabeling are studied.

2 Motivation and New Definitions

A labeling of a graph is assigning labels to the vertices, edges or both vertices and edges. In most applications labels are positive (or nonnegative) integers, though in general real numbers could be used. Magic-type total labeling is assigning the labels to the vertices and edges, such that the sums of labels of all edges incident with the vertex plus the vertex label are all constant. An antimagic total labeling of a graph with p vertices and q edges is a bijection from the set of edges to $1, 2, \ldots, p+q$ such that the sums of the labels of all edges incident with the same vertex plus the vertex label are pairwise distinct. It becomes interesting when we arrive with magic type labeling summing to exactly two distinct constants say k_1 or k_2. Another interesting observation is that for some class of graphs ther exist a bimagic labeling such that the difference between the number of edges with magic constant k_1 and magic constant k_2 seems to be less than or equal to one. This notion leads us to define an induced function for edge cordial labeling. We introduce the following new definitions,

Definition 2.1. *A graph $G(p,q)$ is said to be super edge-even bimagic if there exists a bijection $f : V(G) \cup E(G) \to \{0, 2, \ldots, 2(p+q)\}$ such that $f(u) + f(v) + f(uv) = k_1$ or k_2 for all $e = uv \in E(G)$.*

Definition 2.2. *A graph $G(p,q)$ is said to be super edge-odd bimagic if there exists a bijection $f : V(G) \cup E(G) \to \{1, 3, 5, \ldots, 2(p+q)-1\}$ such that $f(u) + f(v) + f(uv) = k_1$ or k_2 for all $e = uv \in E(G)$.*

Definition 2.3. *If f is a edge bimagic labeling, then we define induced function $g : E \to \{0, 1\}$ for all $e = uv$ such that*

$$g(e) = \begin{cases} 0 & \text{if } f(u) + f(v) + f(uv) = k_1 \\ 1 & \text{if } f(u) + f(v) + f(uv) = k_2 \end{cases}$$

then g is edge bimagic cordial labeling if, $|\eta_0(e) - \eta_1(e)| = 1$ where $\eta_0(e)$ denotes number of edges labeled with 0 and $\eta_1(e)$ denotes number of edges labeled with 1. If f is a super edge Bimagic labeling then the induced function g is said to be a super edge bimagic cordial labeling.

3 General Results on Super Edge Bimagic Labeling

Theorem 3.1. *If G has super edge-magic labeling then, $G + K_1$ has super edge bimagic labeling.*

Proof. Let G be super edge-magic. Then there exist a bijective function $f : V \cup E \to \{1, 2, \ldots, p+q\}$ such that $f(u) + f(uv) + f(v) = k_1$. Now we define the new graph called $G_1 = G + K_1$ with vertex set $V_1(G_1) = V(G) \cup \{x\}$ and $E_1(G_1) = E(G) \cup \{xv_i : 1 \le i \le p\}$. Consider the bijective function

$g : V_1 \cup E_1 \to \{1, 2, \ldots, p+q, p+q+1, p+q+2, \ldots, 2p+q+1\}$ defined by
$g(v_i) = i;\ 1 \leq i \leq p$
$g(v_iv_{i+1}) = f(v_iv_{i+1});\ 1 \leq i \leq p-1;\ 1 \leq j \leq q.$
$g(x) = p + q + 1.$
$g(xv_i) = 2p + q - i + 2;\ 1 \leq i \leq p.$

For the graph G_1 which is super edge magic total,the common count is k_1. Now we have to prove that the another p edges joining $V(G)$ and x yield a common count k_2. For any vertex v_i, $g(x) + g(xv_i) + g(v_i) = p+q+1+2p+q-i+2+1 = 3p+2q+3 = k_2$. Thus we have $G_1 = G+K_1$ has two common counts k_1 and k_2. Hence $G + K_1$ has super edge bimagic labeling. □

Theorem 3.2. *Union of two super edge magic graphs is super edge bimagic.*

Proof. Let G_1 be super edge-magic total. Then there exist the bijective function $f : V_1 \cup E_1 \to \{1, 2, \ldots, p+q\}$ such that $f(u)+f(uv)+f(v) = k_1$ for all $u, v \in V_1$. Let G_2 be the super edge magic. Then there exist the bijective function $g : V_2 \cup E_2 \to \{a+1, \ldots, a+p_1+q_1\}$ such that $g(u)+g(uv)+g(v) = 3a + k_1 = k_2$, for all $u, v \in V_2$, where $a = p+q$. Now we define a new graph $H = G_1 \cup G_2$ with vertex set $V(H) = V_1 \cup V_2$ and edge set $E(H) = E_1 \cup E_2$. Consider the bijective function $h : V \cup E \to \{1, 2, \ldots, a, a+1, \ldots, a+p_1+q_1\}$ defined by $h(u) = f(u)$ for all $u \in V(G_1)$; $h(uv) = f(uv)$, $u, v \in E(G_1)$; $h(v) = g(v)$ for all $v \in V(G_2)$; $h(vw) = g(vw)$, $v, w \in E(G_2)$.

For edges in G_1, we have $h(u)+h(uv)+h(v) = f(u)+f(uv)+f(v) = k_1$. For edges in G_2, we have $h(u) + h(uv) + h(v) = g(u) + g(uv) + g(v) = k_2$.

So $H = G_1 \cup G_2$ has two common count k_1 and k_2. Hence H has super edge bimagic labeling. □

Theorem 3.3. *The graph G has super edge bimagic labeling iff it has both super edge-odd bimagic labeling and super edge-even bimagic labeling.*

Proof. Suppose that G is a graph with p vertices and q edges which has a super edge bimagic labeling. Then there exists a bijective function $h : V(G) \cup E(G) \to \{1, 2, \ldots, p+q\}$ such that $h(u) + h(v) + h(uv) = r_1$ or r_2.

Now we define $f : V(G) \cup E(G) \to \{1, 3, \ldots, 2(p+q) - 1\}$ such that $f(v_i) = 2h(v_i) - 1;\ 1 \leq i \leq p$, $f(e_i) = 2h(e_i) - 1;\ 1 \leq i \leq q$ and $g : V(G) \cup E(G) \to \{0, 2, \ldots, 2(p+q) - 2\}$ such that $g(v_i) = 2h(v_i) - 2;\ 1 \leq i \leq p$, $g(e_i) = 2h(e_i) - 2;\ 1 \leq i \leq q$.Then for any edge $uv \in E(G)$ we have $f(u) + f(v) + f(uv) = 2h(u) - 1 + 2h(v) - 1 + 2h(uv) - 1 = 2[h(u) + h(v) + h(uv)] - 3 = 2(r_1 \text{ or } r_2) - 3$. Thus $k_1 = 2r_1 - 3$ and $k_2 = 2r_2 - 3$. Then G has super edge-odd bimagic labeling with common counts k_1 and k_2. For any edge $uv \in E(G)$ we have, $g(u) + g(v) + g(uv) =$

$2h(u)-2+2h(v)-2+2h(uv)-2 = 2[h(u)+h(v)+h(uv)]-6 = 2(r_1 \text{ or } r_2)-6$. Thus $k_3 = 2r_1 - 6$ and $k_4 = 2r_2 - 6$.

Then G has super edge-even bimagic labeling with common counts k_3 and k_4.

Conversely, suppose that G is a graph with p vertices and q edges and $f : V(G) \in E(G) \to \{1, 3, \ldots, 2(p+q) - 1\}$ is an super edge-odd bimagic labeling with two common counts k_1 and k_2 and $g : V(G) \cup E(G) \to \{0, 2, \ldots, 2(p+q) - 2\}$ is an super edge-even bimagic labeling with two common counts k_3 and k_4. Then $h : V(G) \cup E(G) \to \{1, 2, \ldots, (p+q)\}$ defined by $h(v_i) = 1/4(f(v_i) + g(v_i) + 3)$ for $1 \leq i \leq p$, $h(e_i) = 1/4(f(e_i) + g(e_i) + 3)$ for $1 \leq i \leq q$ is a total edge bimagic labeling with two common edge counts $r_1 = 1/4(k_1 + k_3 + 9)$ and $r_2 = 1/4(k_2 + k_4 + 9)$. □

4 Relationship with Cordial Labeling

Theorem 4.1. *Path graphs P_n, admit super edge bimagic cordial labeling.*

Proof. Let $G(V, E)$ be a path P_n. The vertex set of G is $V(G) = \{v_1, v_2, \ldots, v_n\}$ and the edge set of G is $E(G) = \{v_i v_{i+1}; 1 \leq i \leq n-1\}$. Consider the bijective function $f : V \cup E \to \{1, 2, \ldots, 2n-1\}$ defined by $f(v_i) = i$; $1 \leq i \leq n$ and if n is even then,

$$f(v_j v_{j+1}) = \begin{cases} 2n - 2j + 1; & 1 \leq j \leq n/2 \\ 3n - 2j; & (n/2) + 1 \leq j \leq n - 1 \end{cases}$$

If n is odd then,

$$f(v_k v_{k+1}) = \begin{cases} 2n - 2k + 1; & 1 \leq k \leq (n-1)/2 \\ 3n - 2k - 1; & (n+1)/2 \leq k \leq n - 1 \end{cases}$$

When n is even,
for $1 \leq j \leq n/2$; $f(v_j) + f(v_{j+1}) + f(v_j v_{j+1}) = j + j + 1 + 2n - 2j + 1 = 2n+2 = k_1$ (say) and for $(n/2)+1 \leq j \leq n-1$; $f(v_j)+f(v_{j+1})+f(v_j v_{j+1}) = j + j + 1 + 3n - 2j = 3n + 1 = k_2$ (say)

which concludes that there exist two constants k_1 and k_2 for the path graph P_n. Since the number of vertices is even the number of edges is odd. For $1 \leq j \leq (n/2) + 1$ the number of edges is $n/2$, so it has $n/2$ values of k_1 and for $(n/2) + 1 \leq j \leq n$ the number of edges is $n - 2/2$, so it has $n - 2/2$ values of k_2.
When n is odd,
for $1 \leq k \leq (n-1)/2$; $f(v_k)+f(v_{k+1})+f(v_k v_{k+1}) = k+k+1+2n-2k+1 = 2n + 2 = k_1$ (say) and for $(n+1)/2 \leq k \leq n-1$; $f(v_k) + f(v_{k+1}) +$

$f(v_kv_{k+1}) = k + k + 1 + 3n - 2k - 1 = 3n = k_2$ (say)
which concludes that there exist two constants k_1 and k_2 for the path graph P_n. Since the number of vertices is odd the number of edges is even. For $1 \leq j \leq (n)/2 + 1$ the number of edges is $(n-1)/2$, so it has $(n-1)/2$ values of k_1 and for $(n+1)/2 \leq k \leq n$ the number of edges is $(n-1)/2$, so it has $(n-1)/2$ values of k_2.
Define an induced function $g : E \to \{0, 1\}$ for all $e = uv$ such that

$$g(e) = \begin{cases} 0 & \text{if } f(u) + f(v) + f(uv) = k_1 \\ 1 & \text{if } f(u) + f(v) + f(uv) = k_2 \end{cases}$$

then g admits edge bimagic cordial if $|\eta_0(e) - \eta_1(e)| = 1$ where $\eta_0(e)$ denotes number of edges labeled with 0 and $\eta_1(e)$ denotes number of edges labeled with 1.
When n is even, $|\eta_0(e) - \eta_1(e)| = |(n/2) - ((n-2)/2)| = 1 \leq 1$ then graph G admits edge bimagic cordial.
When n is odd, $|\eta_0(e) - \eta_1(e)| = |((n-1)/2) - ((n-1))| = 0 \leq 1$ then graph G admits edge bimagic cordial.
Since the path P_n is super edge bimagic, it induces super edge bimagic cordial labeling. □

Theorem 4.2. *The Cycle graph C_n, (n is even) admit super edge bimagic cordial labeling.*

Proof. Let $G(V, E)$ be a cycle C_n. The vertex set of G is $V(G) = \{v_1, v_2, v_3, \ldots, v_n$ and the edge set of G is $E(G) = \{v_iv_{i+1}; 1 \leq i \leq n-1\} \cup \{v_nv_1\}$. Consider the bijective function $f : V \cup E \to \{1, 2, \ldots, 2n\}$ defined by $f(v_i) = i$; $1 \leq i \leq n$ and if n is even then

$$f(v_jv_{j+1}) = \begin{cases} 2(n-j+1); & 1 \leq j \leq n/2 \\ 3n - 1 - 2j; & (n/2) + 1 \leq j \leq n-1 \end{cases}$$

$f(v_nv_1) = 2n - 1$.
When n is even, for $1 \leq j \leq n/2$; $f(v_j) + f(v_{j+1}) + f(v_jv_{j+1}) = j + j + 1 + 2n - 2j + 2 = 2n + 3 = k_1$ (say) and for the edge v_nv_1; $f(v_n) + f(v_1) + f(v_nv_1) = n + 1 + 2n - 1 = 3n = k_2$;
also for $(n/2) + 1 \leq j \leq n-1$; $f(v_j) + f(v_{j+1}) + f(v_jv_{j+1}) = j + j + 1 + 3n - 1 - 2j = 3n = k_2$ (say)
which concludes that there exist two constants k_1 and k_2 for the cycle graph C_n. Since the number of vertices is even the number of edges is even.
For $1 \leq j \leq n/2$ the number of edges is $n/2$, so it has $n/2$ values of k_1 and for $(n/2) + 1 \leq j \leq n-1$ and the edge v_nv_1, the number of edges is $n/2$,

so it has $n/2$ values of k_2.
Define an induced function $g : E \to \{0,1\}$ for all $e = uv$ such that

$$g(e) = \begin{cases} 0 & \text{if } f(u) + f(v) + f(uv) = k_1 \\ 1 & \text{if } f(u) + f(v) + f(uv) = k_2 \end{cases}$$

then g admits edge bimagic cordial if $|\eta_0(e) - \eta_1(e)| = 1$ where $\eta_0(e)$ denotes number of edges labeled with 0 and $\eta_1(e)$ denotes number of edges labeled with 1.
When n is even, $|\eta_0(e) - \eta_1(e)| = |(n/2) - (n/2)| = 0 = 1$ then graph G admits edge bimagic cordial .
Since the cycle C_n, (n is even) is super edge bimagic, it induces super edge bimagic cordial labeling. □

Theorem 4.3. *The star graph $K_{1,n}$, admit super edge bimagic cordial labeling.*

Proof. Let $G(V, E)$ be a star graph $K_{1,n}$. The vertex set of G is $V(G) = \{v_1, v_2, v_3, \ldots, v_n, v_{n+1}\}$ and the edge set of G is $E(G) = \{v_1v_i; 2 \leq i \leq n+1\}$. Consider the bijective function $f : V \cup E \to \{1, 2, \ldots, 2n+1\}$ defined by $f(v_i) = i$; $1 \leq i \leq n+1$ and if n is even then

$$f(v_1v_j) = \begin{cases} (3n+6-2j)/2; & 2 \leq j \leq (n+2)/2 \\ (5n+6-2j)/2; & (n+4)/2 \leq j \leq (n+1) \end{cases}$$

If n is odd then

$$f(v_1v_k) = \begin{cases} (3n+7-2k)/2; & 2 \leq k \leq (n+3)/2 \\ (5n+7-2k)/2; & (n+5)/2 \leq k \leq (n+1) \end{cases}$$

when n is even, for $2 \leq j \leq (n+2)/2$; $f(v_1) + f(v_j) + f(v_1v_j) = 1 + j + (3n+6-2j)/2 = (3n+8)/2 = k_1$ (say) and
for $(n+4)/2 \leq j \leq n+1$; $f(v_1)+f(v_j)+f(v_1v_j) = 1+j+(5n+6-2j)/2 = 1 + j + (5n+6-2j)/2 = (5n+8)/2 = k_2$ (say)
which concludes that there exist two constants k_1 and ka_2 for the Star graph $K_{1,n}$. For $2 \leq j \leq (n+2)/2$, the number of edges connected to v_1 is $n/2$, so it has $n/2$ values of k_1 and for $(n+4)/2 = j = n+1$ the number of edges connected to v_1 is $n/2$, so it has $n/2$ values of k_2.

When n is odd, for $2 \leq k \leq (n+3)/2$; $f(v_1) + f(v_k) + f(v_1v_k) = 1+k+(3n+7-2k)/2 = (3n+9)/2 = k_1$ (say) and for $(n+5)/2 \leq k \leq n+1$; $f(v_1) + f(v_k) + f(v_1v_k) = 1 + k + (5n+7-2k)/2 = (5n+9)/2 = k_2$ (say) which concludes that there exist two constants k_1 and k_2 for the Star graph $K_{1,n}$. For $2 \leq k \leq (n+3)/2$ the number of edges connected to v_1 is $(n+1)/2$, so it has $(n+1)/2$ values of k_1 and for $(n+5)/2 \leq k \leq n+1$ the number of edges connected to v_1 is $(n-1)/2$, so it has $(n-1)/2$ values of k_2.

Define an induced function $g: E \to \{0,1\}$ for all $e = uv$ such that

$$g(e) = \begin{cases} 0 & \text{if } f(u) + f(v) + f(uv) = k_1 \\ 1 & \text{if } f(u) + f(v) + f(uv) = k_2 \end{cases}$$

then g admits edge bimagic cordial if $|\eta_0(e) - \eta_1(e)| = 1$ where $\eta_0(e)$ denotes number of edges labeled with 0 and $\eta_1(e)$ denotes number of edges labeled with 1.
When n is even, $|\eta_0(e) - \eta_1(e)| = |(n/2) - (n/2)| = 0 \leq 1$ then graph G admits edge bimagic cordial.
When n is odd, $|\eta_0(e) - \eta_1(e)| = |((n+1)/2) - ((n-1)/2)| = 1 \leq 1$ then graph G admits edge bimagic cordial.
since the star graph $K_{1,n}$ is super edge bimagic,it induces super edge bimagic cordial labeling.

5 Conclusion

Edge bimaic total labeling for a graph is not unique. For example there are exactly $14(n-1)$ ways of giving an edge bimagic total labeling for $K_{1,n}$ [9]. But investigation shows that there may be only a unique way of super edge bimagic labeling that partitions the edge set into two sets with almost equal edges that induces super edge bimagic cordial labeling.

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