Relative Character Graph using Group Representation

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Abstract: In this paper we introduce a finite graph using group characters and discuss the basic properties of the graph.

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1. Introduction

Construction of finite graphs using characters of finite groups is not a new theme. In the early 1940's, Richard Brauer, one of the pioneers of modular representations, constructed a graph using modular characters, which were later called 'Brauer Graphs'. [2] Of late, people working on representations of algebraic groups and related finite groups extensively use Brauer Trees. Recently, new finite graphs were constructed using group characters by Chigira and Iyori [3], Manz [17], Willems [16] et al.

2. The Relative Character Graph Γ (G, H) background from Group Representations

We now proceed to describe the graph Γ (G, H) with a brief introduction to the representation (character) theory of finite groups. We refer to the treatises of Curtis and Reiner [5] and Isaacs [8] for definitions, notations and properties.

2.1. Definition

Let V be an n dimensional vector space over a field F, and let G be a finite group. A representation of G with vector space V is a homomorphism ρ

of G into the group GL(V), the group of invertible linear transformations of V. The dimension n is called the degree of ρ.

Choosing a suitable basis of V, we can convert each element of ρ(s), s$\in$ G, into GL (n, F) and the resulting homomorphism is called the (matrix) representation of G associated to ρ. (Of course, change of basis of V results in a different matrix, but the original and the new matrices are similar. This crucial fact will be used when we pass on to characters).

We use the same notation ρ for the representation through the space V or through matrices, it is sufficient to take F as the complex field C, which is algebraically closed and has characteristic 0. Such representations are called 'ordinary representations' in the literature. (The latest terminology is 'non-describing'.) The homomorphism which takes every element of G into the complex number1 is called the trivial representation and is usually denoted by 1_G.

2.2. Definition

A subspace W of V is said to be G- invariant (or ρ - invariant or simply invariant) if ρ (s) (w) $\in$ W for all w$\in$ W and s$\in$ G.

2.3. Definition

A representation (ρ ,V) is irreducible if (0) and W are the only invariant subspaces of V. Otherwise, it is said to be reducible.

2.4. Theorem(*Maschke*) (see e.g. [8])

Every representation of G is a (finite) direct sum of irreducible representations of G.

2.5. Definition

Let ρ be a (matrix) representation of G. Then the character χ_ρ of G (or simply $\chi, \chi: G \to F$ given by $\chi(s) = Trace\ (\rho)$. whereas χ is not a homomorphism, it has the excellent property that $\chi(tst^{-1}) = \chi(s)$ for all $s, t \in G$ (ie, χ is invariant on the conjugate classes of G). Further two representations of G are equivalent if and only if they have the same character (in our complex case, or, more generally, when the characteristic of F does not divide O(G)). This enables us to pass on from representations into characters.

2.6. Definition

If ϕ and ψ are two characters of G, then the scalar product is defined as $(\phi,\psi)=\frac{1}{O(G)}\sum_{s\in G}\phi(s)\overline{\psi}(s)$

2.7. Properties

1. ϕ is irreducible if and only if $(\phi,\phi)=1$.

2. The number of inequivalent (distinct) irreducible characters (characters of irreducible representations) is finite and this number is to the number of conjugacy classes of G.

3. If $\chi_1, \chi_2, \chi_3, \ldots\ldots, \chi_h$ are the distinct irreducible characters of G with degrees $n_1, n_2, n_3, \ldots\ldots, n_h$ respectively, then,

(i) $n_1^2, n_2^2, n_3^2, \ldots\ldots\ldots n_h^2 = O(G)$. (ii) n_i divides O(G) for every i.

3. Restriction and Induction

Let ρ be a representation of G with character χ and let H be a subgroup of G. Then the restriction $\rho_H : H \to GL(n,C)$ is clearly a representation of H and we shall denote its character by χ_H (Note that χ is irreducible for G need not imply χ_H is irreducible for H).

Let θ be a character of H. Then the induced character θ^G for G is defined as

$$\theta^G(g)=\frac{1}{O(H)}\sum_{x\in G}\theta^0(xgx^{-1}),\qquad x\in G,$$

When $\theta^0(h) = \theta(h), h \in H$ and

=0 otherwise

These two concepts play a crucial role in our graph construction.

2.8. Notation
The set of all distinct irreducible characters of G is denoted by *Irr G* and the corresponding set for H is denoted by *Irr* H.(Apart from the trivial character it is very rare that these two sets intersect.)

2.9. Theorem (Frobenius Reciprocity Formula)
Let H be a subgroup of G, let $\theta \in Irr H$ and let $\chi \in Irr G$. Then $(\chi_H, \theta)_H = (\chi, \theta^G)_G$.

2.10.Definition
Let H be normal subgroup of G and let θ be any character of H. Then the conjugate θ^s of θ for any character of H is defined as $\theta^s(h) = \theta(shs^{-1})$. It is known that θ^s is irreducible if and only if θ is irreducible.

2.10. Theorem (Clifford)
Let H be a normal subgroup of G and let $\chi \in Irr G$. Let θ be an irreducible constituent of χ_H and suppose $\theta = \theta_1, \theta_2, \ldots\ldots\ldots \theta_t$ are the distinct conjugates of θ in G. Then $\chi_H = e\sum_{i=1}^{t}\theta_i$ where $e = (\theta, \chi_H)$. That is, each θ_i does occur and with the same multiplicity in χ_H.

4. Relative Character Graph Γ(G,H)

4.1.Definition
Let H be an arbitrary subgroup of G. Then the Relative Character Graph Γ(G,H) of G with respect to H has the elements of IrrG as vertices and two distinct vertices (character) χ and ψ are adjacent if and only if χ_H

and ψ_H have atleast one element θ of IrrH in common. This is equivalent to saying that $(\chi,\psi)_H > 0$. Clearly Γ(G,H) is a simple graph.

The following are simple observations.

I. Γ(G,H) is the null graph if and only if H=G.
II. If K is a subgroup of G contained in H, then Γ(G,H) is a subgraph of Γ(G,K).
III. Where Γ(G,(1)) is complete ((1) denotes the trivial subgroup), the converse is not true. In fact, if H is a cyclic subgroup of order r generated by x and if each ρ_i(x) has 1 as eigen value, where ρ_i runs through the full set of irreducible representations of G, then Γ(G,H) is complete.

Our next aim is to prove a useful criterion for the connectivity of Γ(G,H). We can then see that the same method can be generalized to get the connected components of Γ(G,H).

First, we shall note that the adjacency stipulation in the definition of Γ(G,H) is equivalent to the following revised formulation.

It is a well-known fact that G acts on the set of right cosets of H in G, denoted by G/H. This action is clearly transitive and this gives a representation of G called the permutation representation of G acting on G/H. It is known that the character of this representation is nothing but the induced character 1_H^G, where 1_H denotes the trivial character of H. (For details, we refer to Isaac's book [8].)Set $1_H^G = \chi$. Then we can immediately prove the following

4.2.Proposition

Two vertices ϕ and ψ are adjacent in Γ(G,H) if and only if ψ is a constituent of $\phi\chi$

Proof: First note that $\phi_H^G = (\phi_H 1_H)^G = \phi 1_H^G$ (by a property of induced characters) $= \phi\chi$. Therefore, ψ is an irreducible constituent of $\phi\chi$ if and only if $(\phi\chi,\psi)_G = (\phi_H^G,\psi)_G \neq 0$ and hence $(\theta^G,\psi)_G \neq 0$ for some H-irreducible constituent θ of ϕ_H . Hence $(\theta, \psi_H)_H \neq 0$ (by Frobenius reciprocity formula), which is the original adjacency criterion.

4.3.Lemma

An element φ of IrrG (=vertex set of Γ(G,H))is joined to 1_G by a path of length s if and only if φ is an irreducible constituent of χ^s .

Proof: We apply induction on s. This is clear for s = 0 and s = 1. Assume that s >1 and the lemma holds for s-1. Suppose that φ is joined to 1_G by a path of length s, but not of length s-1. Let $1_G = \varphi_0, \varphi_1, \ldots, \varphi_s = \varphi$ be this path. By inductive assumption, φ_{s-1} is an irreducible constituent of χ^{s-1} and since φ_{s-1} and φ are adjacent, φ is an irreducible constituent of $\varphi_{s-1}\chi$(by the above proposition), and hence clearly φ is an irreducible constituent of χ^s .

Conversely, assume that φ is an irreducible constituent of χ^s . Then for some irreducible constituent ψ of χ^{s-1} , φ is an irreducible constituent of $\psi\chi$. Thus φ and ψ are adjacent. But by the induction assumption, ψ is joined to 1_G by a path of length s-1, and hence there is a path of length s from φ to 1_G . (We shall call this lemma as the path lemma hence forth). This is only an algorithmic lemma which just works for all s $\leq$ IrrG.

4.4.Definition

For any subgroup H of G,

$Core_G H = \bigcap_{g \in G} gHg^{-1}$, which is the largest normal subgroup of G contained in H. If we take $\Omega = G/H$, then $Core_G H = \{x \in G$: wx = w for all w$\in \Omega\}$

$= \ker \chi$. (Kernel of the corresponding representation)

4.5.Proposition

If $\Gamma_1(G,H)$ is the connected component of Γ(G,H) containing 1_G,

then i) $Core_G H = \bigcap_{\phi \in \Gamma_1(G,H)} \ker \phi$

ii) $\Gamma_1(G,H)$ is the set of irreducible characters of G with Kernel containing $Core_G H$.

Proof: i) Let N = $Core_G H$. Since $\Gamma_1(G,N)$ is a complete subgraph containing 1_G and $\Gamma_1(G,H)$ is a subgraph of $\Gamma_1(G,N)$, it is clear that $H \leq \bigcap_{\phi \in \Gamma_1(G,H)} \ker \phi$ Now, let $\lambda_1, \lambda_2, \ldots\ldots, \lambda_m$ be the irreducible constituents of χ. Then

$$\bigcap_{\phi \in \Gamma_1(G,H)} \ker \phi \subseteq \bigcap_{i=1}^{m} \ker \lambda_i = \ker \chi = N$$ Hence we have

$$N = \bigcap_{\phi \in \Gamma_1(G,H)} \ker \phi$$

ii) Let $\Gamma_1^*(G,H)$ denote the set of all irreducible characters with kernel containing N = $Core_G H$ .Then by (i) $\Gamma_1(G,H)$ = $\Gamma_1^*(G,H)$, since if $\phi \in \Gamma_1(G,H)$, then

$\ker \varphi \supseteq \bigcap_{\varphi \in \Gamma_1(G,H)} \ker \varphi = N$,which implies $\varphi \in \Gamma_1^*(G,H)$.

Consider χ and ψ as characters of G/N. Since χ is faithful, Brauer-Burnside theorem, φ is an irreducible constituent of χ^r for some $r \geq 1$. Hence by the above lemma, φ is connected to 1_G. That is $\varphi \in \Gamma_1(G,H)$. Hence $\Gamma_1(G,H) = \Gamma_1^*(G,H)$, which proves the lemma.

4.6.Corollary

$\Gamma_1(G,H) = \Gamma(G,H)$ if and only if $Core_G H = (1)$.

In other words, the corollary says that Γ(G,H) is connected if and only if $Core_G H = (1)$.

4.7.Corollary

If G is simple group and H is any proper subgroup, then Γ(G,H) is connected.

Proof: Just observe that $Core_G H = (1)$ since G is simple. We can slightly modify lemma in the following manner, which gives explicitly all the connected components of Γ(G,H). We omit the proof.

4.8.Theorem

Two vertices φ and ψ lie in the same component if and only if φ is a constituent of $\psi\chi^s$.

5. The Tree Problem

From now on, we shall quickly develop the theory, sometimes omitting details of the proofs, as the techniques and idea of the proofs are already indicated through proofs of results that we have described so far. Before going into the tree problem itself, we recall the concept of a "Frobenius group".

5.1.Definition

G is a Frobenius group if there is a non-trivial subgroup H such that $H \cap H^x = (1)$ for all $x \notin H$,where $H^x = xHx^{-1}$. In this case there is a distinguished normal subgroup N such that G is the semi direct product of N and H. The subgraphs N and H are called the Frobenius Kernel and Frobenius complement respectively. For example S_3, A_4 and D_{2n},the Dihedral group of 2n elements when n is odd are Frobenius groups.

By the very nature of the subgroup H, the graphs Γ(G,H)is connected if G is a Frobenius group with complement H. In fact we can say much more.

5.2.Theorem

The graph Γ(G,H)is connected, then it is a tree if and only if G is a Frobenius group with complement H such that the Frobenius Kernel N is the unique minimal elementary abelian normal Sylow p-subgroup of order p^m, for some prime p, and O(H) = p^m-1. For details we refer to [6].

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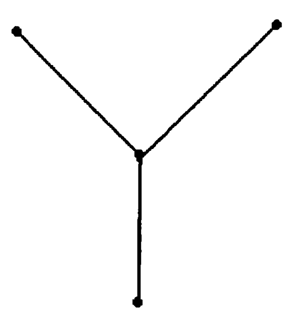
5.3.Example

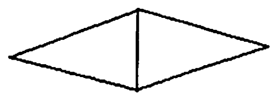
The alternating graph A_4 is a semi direct product of N, the Klein 4-group of order 4 and H, the cyclic group of order3. The graph Γ(G,H) is the following

1_G



However the dihedral group D_{10}, although Frobenius where the complement H is cyclic group of 2 elements and the Kernel N is the cyclic graph of 5 elements, but $2 \neq 5-1$ even though 2 divides 5-1 (according to basic properties). The graph $\Gamma(D_{10},H)$ drawn below is clearly not a tree.

1_G



In fact, if $\Gamma(G,H)$ is a tree then it must be isomorphic of the star $K_{1,n}$. The central vertex Λ is very important, because the tree situation forces the right action of G on G/H to be doubly transitive which is equivalent to $1_H^G = 1_G + \Lambda$.

6. Triangulations

Triangulation problems, apart from its intrinsic interest in graph theory, are nowadays widely used in "Global Positioning Systems" in communication engineering.

Recall that a (connected) graph in which each cycle of length at least four has a chord is called a *triangulated graph*

The following development leading to our main theorem is generally based on Parthasarathy's book [12]. For a vertex v in an arbitrary graph Γ, N(v) denotes the set of vertices adjacent with v and <N(v)> denotes the subgraph induced by N(v).

6.1.Definition

A vertex v is simplicial if <N(v)> is a complete graph. If $\sigma = \{v_1, v_2, \ldots v_n\}$ is an ordering of the vertices of Γ such that $1 \leq i \leq n$, v_i is a simplicial vertex of the induced subgraph

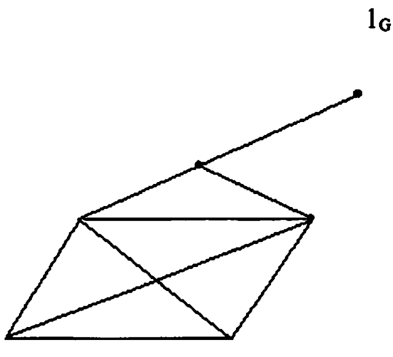
$\Gamma_i = \langle v_i, v_{i+1}, \ldots v_n \rangle$, then σ is called a *perfect elimination scheme* for Γ.

6.2.Theorem[12]

Γ is triangulated if and only if it has a perfect elimination scheme. We state, without proof, our main.

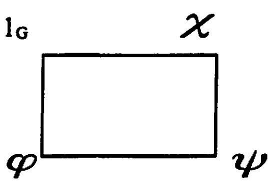
6.3.Theorem

If $\Gamma(G,H)$ is connected for any pair (G,H) then $\Gamma(G,H)$ has a perfect elimination scheme.(Note that we include the trees also here, under the assumption that any tree is trivially triangulated). A typical example is the following: Consider G = PSL(2,7), the quotient group by its center of the special linear group SL(2,7) of 2x2 matrices with determinant 1 over a field of 7 elements and let H = the subgroup S_4 sitting inside G. $\Gamma(G,H)$ is drawn below:



6.4.Remark

Not every finite graph can be isomorphic to $\Gamma(G,H)$ for some group G and subgroup H. For instance, the graph



Can not be Γ(G,H) for any group G and subgroup H. This can be quickly seen as follows: The edge $1_G \phi$ gives the information, hat 1_H occurs in φ_H . Similarly 1_H occurs in χ_H . Hence by definition, φ and χ themselves are forced to be adjacent which is contradiction.

7. COMPLEMENTS

Finally let us consider the complement situation of Γ(G,H) . We have the following interesting result perhaps highlighting graph theoretically as to why the subgroup H is called a 'complement' in a Frobenius group NH, with Kernel N.

7.1.Theorem

G=NH is a semi direct product with N normal and H non-normal, then G is Frobenius with Kernel N and complement H if and only if $\Gamma(G,H) = \overline{\Gamma}(G,H)$. We record the following further facts on these lines. It is known in graph theory that if a graph Γ is disconnected, then $\overline{\Gamma}$ is always connected. But when Γ is connected $\overline{\Gamma}$ can be either disconnected or connected. In this context the following theorem is very interesting. For details of the proof we refer the reader to [6].

7.2.Theorem

Let n denote the number of vertices of Γ(G,H) and q, the number of edges. Suppose Γ(G,H) is not a tree and the right action of G on G/H is doubly transitive. If further $q \leq n-1C_2$, then $\overline{\Gamma}$(G,H)is connected. There is one more interesting result giving a criterion for the connectedness of $\overline{\Gamma}$ (G,H) using diameters.

7.3.Theorem

Let $Core_G H = (1)$ and let the right action of G on G/H is doubly transitive. Then $\overline{\Gamma}$(G,H)is connected if and only if the diameter of Γ(G,H)is atleast 3.

8.Conclusion

The graph B(G,H) of Chigira and Iyori[3] is different from Γ(G,H). Whereas B(G,H) is a bipartite graph, Γ(G,H) has no such restrictions, even though the (topological) property of connectedness criterion is the same in both the graphs.

We have just opened up a way to construct finite graphs using character theory of finite groups. Many interesting questions can be raised and the answers to these questions will definitely enrich group theory, character theory as well as graph theory.

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