On the Radio Number of the Hexagonal Mesh

Bharathi Rajan and Kins Yenoke
Department of Mathematics, Loyola College, Chennai 600 034, India.
Email: jecinthokins@rediff.com

Abstract

A radio labeling of a connected graph G is an injection f from the vertices of G to the natural numbers such that $d(u,v) + |f(u) - f(v)| \geq 1 + diam(G)$ for every pair of distinct vertices u and v of G. The radio number of f denoted $rn(f)$, is the maximum number assigned to any vertex of G. The radio number of G, denoted $rn(G)$, is the minimum value of $rn(f)$ taken over all labelings f of G. In this paper we determine bounds for the radio number of the hexagonal mesh.

Keywords: labeling, radio labeling, radio number and hexagonal mesh.

1 Introduction

Interest in graph labeling problems began in the mid-1960's with a conjecture of Ringel[19] and a paper by Rosa [21]. In the intervening years dozens of graph labelings techniques have been studied in over several papers. Despite the large number of papers, there are relatively few general results or methods on graph labeling. Indeed most of the results focus on particular classes of graphs and utilize adhoc methods. Frequently, the same classes have been done by several authors. Labeled graphs serve as useful models for a broad range of applications such as coding theory, x-ray, crystallography, radar, astronomy, circuit design, channel assignments of FM radio stations and communication network addressing applications [3,4].

In a direct interconnection network, nodes represent processors, while edges indicate connections between processors for direct message exchange. Some interconnection network topologies are designed and some borrow from nature. Hexagonal mesh is one of such natural architecture. Hexagonal mesh are based on regular triangular tessellations.

2 An Overview of the Paper

The use of graph theory to study the Channel Assignment Problem and related problems dates back at least to 1970 (see [18]). In 1980, Hale [11] modeled the Channel Assignment Problem as both a frequency-distance constrained and frequency constrained optimization problem and discussed applications to important real world problems. Since then, a number of graph colorings have been inspired by the Channel Assignment Problem.

In 2001 Chartrand et al. [5] were motivated by regulations for channel assignments of FM radio stations to introduce radio labelings of graphs. Two stations that share the same channel must be separated by at least 115 kilometers; however, the actual required separation depends on the classes of the two stations. Two channels are considered to be first-adjacent (or simply adjacent) if their frequencies differ by 200 kHz, that is, if they are consecutive on the FM dial. For example, two FM stations on channels 92.4 MHz and 92.6 MHz are adjacent. The distance between two radio stations on adjacent channels must be at least 72 kilometers. Again, the actual restriction depends on the classes of the stations. The distance between two radio stations whose channels differ by 400 or 600 kHz (second- or third-adjacent channels) must be at least 31 kilometers. Once again, the actual required separation depends on the classes of the stations.

For a connected graph G of diameter d and an integer k with $1 \leq k \leq d$, a k-radio coloring (sometimes called a radio k-coloring) of G is an assignment f of colors (positive integers) to the vertices of G such that $d(u,v) + |f(u) - f(v)| \geq 1 + k$ for every two distinct vertices u and v of G. A *radio labeling* of a connected graph G is an injection f from the vertices of G to the natural numbers such that $d(u,v) + |f(u) - f(v)| \geq 1 + diam(G)$ for every two distinct vertices u and v of G. *The radio number of* f, denoted by $rn(f)$, is the maximum number assigned to any vertex of G. The *radio number of* G, denoted by $rn(G)$, is the minimum value of $rn(f)$ taken over all labelings f of G.

The radio numbers for different families of graphs have been studied by several authors. For paths and cycles, the radio number problem has been studied by Chartrand et al. [7] and by Zhang [24] and completely determined by Liu and Zhu [17]. The radio number for square of paths and of cycles was investigated by Liu et al. [16]. Bharati et al. [1] determined the radio number of wheels, fans, double fans, Dutch-t-mill graph, star graph, uniform$(r + k - 1)$-cyclic split graph and uniform r-cyclic star split graphs. More recently Khennoufa et al.[13] have investigated the radio number of hypercube.

In this paper we have obtained a general result pertaining to the lower bound of the radio number of a connected graph. We also determined the bounds for the radio number of hexagonal mesh.

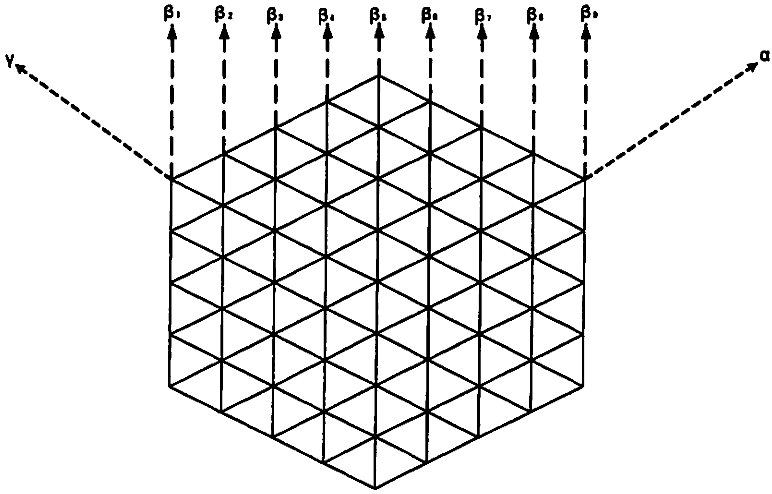

Figure 1: β-lines in HX_5

3 The Radio Number of Hexagonal Mesh

The triangular tessellation is used to define a hexagonal mesh and this is widely studied in [8, 23]. Hexagonal torus was used in HARTS project [23]. We denote the hexagonal mesh of dimension n by HX_n.The hexagonal mesh HXn has $3n^2 - 3n + 1$ vertices and $9n^2 - 15n + 6$ edges, where n is the number of vertices on one side of the network [8]. There are six vertices of degree three which we call as corner vertices. There is exactly one vertex v at distance $n - 1$ from each of the corner vertices. This vertex is called the centre of HXn. Let α, β and γ be the three axes inclined mutually at an angle 60° respectively and let $\beta_1, \beta_2 \ldots \beta_{2n-1}$ be the β lines (vertical lines) marked from left to right as shown in the figure 1. We name vertices on $\beta_1, \beta_2 \ldots \beta_{2n-1}$ lines as follows: We name the vertices on β_1 from top to bottom as $v_1, v_2 \ldots v_n$, the vertices on β_2 from top to bottom as $v_{n+1}, v_{n+2} \ldots v_{2n+1}$. Finally we name the vertices on β_{2n-1} from top to bottom as $v_{3n^2-4n+2}, v_{3n^2-4n+3} \ldots v_{3n^2-3n+1}$. The diameter of HX_n is $2n - 2$. We first provide an upper bound for the radio number of the hexagonal mesh of dimension n.

We begin with certain definitions.

The *geodesic distance* or *distance* between two vertices in a graph is the number of edges in a shortest path connecting them.

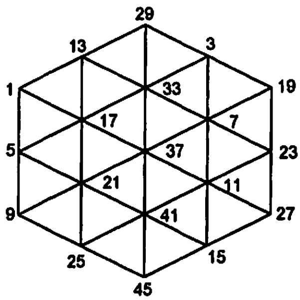

Figure 2: A radio labeling of HX_3 which gives an upper bound.

The *eccentricity* $e(v)$ of a vertex v is the greatest geodesic distance between v and any other vertex in the graph G. That is,

$$e(v) = max\{d(v, w) \forall w \in V(G)\}.$$

The *radius* of G is the minimum eccentricity among the vertices of G and the *diameter* of G is the maximum eccentricity among the vertices of G.

The *center* of G is the set of vertices of eccentricity equal to the radius.

For any connected graph G, $diam(G) \geq e(v) \geq rad(G)$ for every vertex v in G.

Theorem 1 *Let G be HX_n. Then the radio number of G satisfies $rn(G) \leq n(3n^2 - 4n - 1) + 3$.*

Proof. Define a mapping $f\colon V(G) \longrightarrow N$ as follows:

$$f(v_i) = 2(n-1)(i-1) + 1; i = 1, 2 \ \dots \ \frac{n}{2}(3n-1),$$

$$f(v_{\frac{n}{2}(3n-1)+i}) = (n-1)(2i-1) + 1; i = 1, 2 \ \dots \ \frac{n}{2}(3n-5) + 1.$$

We claim that $d(u, w) + |f(u) - f(w)| \geq 2n - 1$ for all $u, w \in V(G)$.

Case(i): $u, w \in V(\beta_i), 1 \leq i \leq n$.

Let $u = v_l$ and $w = v_k, 1 \leq l, k \leq \frac{n}{2}(3n-1), l \neq k$. Then $d(u, w) \geq 1$ and $d(u, w) + |f(u) - f(w)| \geq 1 + |2(n-1)(l-1) + 1 - (2(n-1)(k-1) + 1)| = 1 + |2(n-1)(l-k)| \geq 2n - 1$.

Case(ii): $u \in V(\beta_i), w \in V(\beta_j), 1 \leq i, j \leq n, i \neq j$

Let $u = v_l$ and $w = v_k, 1 \leq l, k \leq \frac{n}{2}(3n-1)$, $l \neq k$. Then as in Case (i), $d(u, w) + |f(u) - f(w)| \geq 2n - 1$.

Case(*iii*): $u, w \in V(\beta_i)$, $n+1 \le i \le 2n-1$.

Let $u = v_{(\frac{n}{2}(3n-1)+l)}$ and $w = v_{(\frac{n}{2}(3n-1)+k)}$, $1 \le l, k \le \frac{n}{2}(3n-5)+1$, $l \ne k$. Then $d(u,w)+|f(u)-f(w)| \ge 1+|(n-1)(2l-1)+1-((n-1)(2k-1)+1)| = 1+|2(n-1)(l-k)| \ge 2n-1$.

Case(*iv*): $u \in V(\beta_i), w \in V(\beta_j)$, $n+1 \le i, j \le 2n-1$.

Let $u = v_{(\frac{n}{2}(3n-1)+l)}$ and $w = v_{(\frac{n}{2}(3n-1)+k)}$, $1 \le l, k \le \frac{n}{2}(3n-5)+1$, $l \ne k$. Then $d(u,w) \ge 1$ and

$d(u,w)+|f(u)-f(w)| \ge 1+|(n-1)(2l-1)+1-((n-1)(2k-1)+1)| \ge 2n-1$.

Case (*v*): $u \in V(\beta_i), 1 \le i \le n$ and $w \in V(\beta_j)$, $n+1 \le j \le 2n-1$.

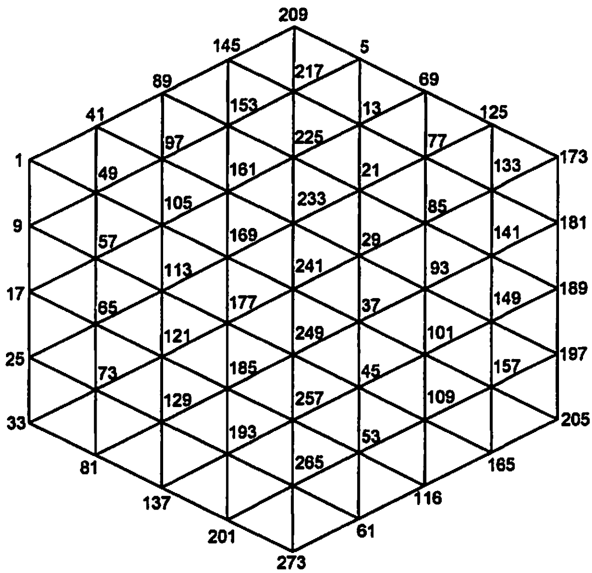

Figure 3: A radio labeling of HX_5 which gives an upper bound.

Subcase (*a*): Suppose $u \notin (V(\beta_j) \cup V(\beta_{j+1}))$ and $w \notin V(\beta_{n+j}), j = 1, 2 \ldots n-1$, then $d(u,w) \ge 1$ and $|f(u)-f(w)| \ge 7(n-1)$. Now $d(u,w) + |f(u)-f(w)| \ge 1+7(n-1) > 2n-1$.

Subcase(*b*): If $u \in V(\beta_j)$ and $w \in V(\beta_{n+j}), 1 \le j \le n-1$ then $u = v_l, 1 \le l \le \frac{n}{2}(3n-1)$ and $w = v_{(\frac{n}{2}(3n-1)+k)}$, $1 \le k \le \frac{n}{2}(3n-5)+1$. Therefore $d(u,w) = n$, and $d(u,w)+|f(u)-f(w)| \ge n+|2(n-1)(l-1)+1-((n-1)(2k-1)+1)| = n+|(n-1)(2(l-k)-1)| \ge 2n-1$.

Subcase (*c*): Suppose $u \in V(\beta_n)$ and $w \in V(\beta_{2n-1})$ then either $d(u,w) \ge$

$n-1$ and $|f(u)-f(w)| \geq 3n-3$ or $d(u,w) = 2n-2$ and $|f(u)-f(w)| = n-1$. In both the cases, $d(u,w) + |f(u) - f(w)| \geq 2n - 1$

Subcase (d): If $u \in V(\beta_{j+1}), w \in V(\beta_{n+j}), 1 \leq j \leq n$ then either $d(u,w) = n$ and $|f(u)-f(w)| = n-1$ or $d(u,w) = n-1$ and $|f(u)-f(w)| = n$. In both the cases, $d(u,w) + |f(u) - f(w)| \geq 2n - 1$. Thus f is a radio labeling. Hence $rn(G) \leq n(3n^2 - 4n - 1) + 3$. □

Theorem 2 *Let G be a simple connected graph of order n. Let $n_0, n_1 \ldots n_k$ be the numbers of vertices having eccentricities $e_0, e_1 \ldots e_k$, where $diam(G) = e_0 > e_1 > \cdots > e_k = rad(G)$. Then*

$$rn(G) \geq \begin{cases} n - 2(d-e_k) + \sum_{i=1}^{k} 2(d-e_i)n_i, & \text{if } n_k > 1 \\ n - (d-e_k) - (d-e_{k-1}) + \sum_{i=1}^{k} 2(d-e_i)n_i, & \text{if } n_k = 1 \end{cases}$$

Proof. Let v_l be a vertex with eccentricity $e_l = d - l, 0 \leq l \leq k$. Then there exists a vertex w in G such that $d(v_l, w) = d - l$. By the definition of radio labeling, the modulus of difference in function values of v_l and w must be at least $l + 1$. Suppose we give a label for w as 1, then the label for v_l must be at least $l + 2$. In order to achieve the radio number we label the next vertex with eccentricity e_l. Let u be a vertex with eccenctricity e_l. The label of u is at least $2l + 3$.Hence at least $2l$ natural numbers are skipped in the radio labeling of vertices with eccentricity e_l, other than the first and last radio labeled vertex. Consider the n_k vertices $x_1, x_2 \ldots x_{n_k}$ with minimum eccenctricity e_k.To obtain the radio number we start and finish labeling with these vertices. Without loss of generality we may start with x_1 and finish with x_{n_k}. Needless to say if $n_k > 1$, then $2(d - e_k)$ natural numbers are skipped while labeling x_1 and x_{n_k} of G. If $n_k = 1$, we start with x_1 and end with a vertex say y_1 with eccenctricity e_{k-1}. Since there are $n_0, n_1 \ldots n_k$ number of vertices with eccentricities $e_0, e_1 \ldots e_k$, where $diam(G) = e_0 > e_1 > \cdots > e_k = rad(G)$, at least $\sum_{i=1}^{k} 2(d-e_i)n_i - 2(d-e_k)$ or $\sum_{i=1}^{k} 2(d-e_i)n_i - (d-e_k) - (d-e_{k-1})$ numbers are skipped in the radio labeling of vertices of G, according as $n_k > 1$ or $n_k = 1$. Since G

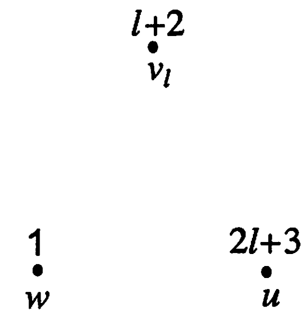

Figure 4: Radio labeling of a vertex with eccenctricity e_l.

is of order n, the radio number of G is at least

$$rn(G) \geq \begin{cases} n - 2(d - e_k) + \sum_{i=1}^{k} 2(d - e_i)n_i, \text{ if } n_k > 1 \\ n - (d - e_k) - (d - e_{k-1}) \ + \sum_{i=1}^{k} \ 2(d - e_i)n_i, \text{ if } n_k = 1 \end{cases} . \square$$

Remarks: (i) The graph in figure 5 attains the bound given in theorem 2, thereby showing that the bound is sharp.

(ii) In the case of HX_n the eccenctricities are given by $e_0 = 2n-2, e_1 = 2n-3, \ldots, e_{n-2} = n$, $e_{n-1} = n-1$. Therefore $diam(HX_n) = 2n-2$ and $rad(HX_n) = n-1$

(iii) The number of vertices having eccentricities e_{n-1} is 1, e_{n-2} is $6, \ldots,$ e_1 is $6(n-2)$, e_0 is $6(n-1)$.

Theorem 3 *Let HX_n be a Hexagonal mesh of dimension n and let $e_0, e_1 \ldots e_{n-1}$ be the eccenctrites of the vertices of the Hexagonal mesh such that $e_0 > e_1 > \ \ldots \ > e_{n-1}$, then*

$$rn(HX_n) \geq 3n^2 - 3n + \quad 12\sum_{i=0}^{n-1} i(n-i-1), if \ n > 1.$$

Proof. Since there are $6(n-i-1), i = 0, 1 \ \ldots \ n-2$ vertices having eccenctricities e_i, $i = 0, 1, 2 \ \ldots \ n-2$ and a single vertex with eccenctricity e_{n-1}, by theorem 2, we have,

$$rn(HX_n) \geq 3n^2 - 3n + 1 + n - 1 - n - 2 + \sum_{i=0}^{n-1} 2.6i(n-i-1)$$

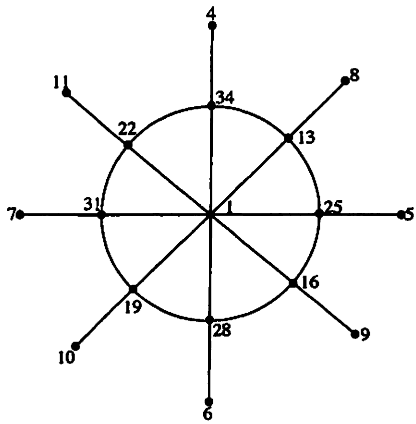

Figure 5: A radio labeling of Helm H_n which attains the lower bound.

$$= 3n^2 - 3n + \quad 12\sum_{i=0}^{n-1} i(n-i-1), if n > 1.$$

4 Conclusion

In this paper we have obtained a general result giving a lower bound for the radio number of a connected graph. We have applied the same to the hexagonal mesh. The radio number problem for Torus mesh, Honeycomb mesh, Silicate network etc., are under investigation.

References

[1] Bharati Rajan, Indra Rajasingh, Kins Yenoke, Paul Manuel, "*Radio number of graphs with small diameter*", International Journal of Mathematics and Computer Science, Vol.2, pages 209-220, 2007.

[2] Bharati Rajan, Indra Rajasingh, Kins Yenoke, "*Radio number of wheel related graphs*", Proceedings of the International conference of Mathematics and Computer Science, Vol.1, pages 178-182, 2009.

[3] G.S. Bloom, S.W. Golomb, "*Application of undirected graphs*", Proc. IEEE, Vol.65, pages 562-570, 1977.

[4] G. S.Bloom, S.W. Golomb, "*Numbered complete graphs, unusual rulers, and assorted applications*", Theory and applications of graphs, Lecture notes in Mathematics 642. Springer-Verlag, New york, pages 53-65, 1978.

[5] G. Chartrand, D. Erwin, P. Zhang, "*A radio labeling problem suggested by FM channel restrictions*", preprint, 2001.

[6] G. Chartrand, D. Erwin, P. Zhang and F. Harary, "*Radio labelings of graphs*", Bull.Inst.Combin. Appl., Vol 33, pages 77-85, 2001.

[7] G. Chartrand, L. Nebesk´y and P. Zhang, "*Radio $k-$colorings of paths*", Discussiones Mathematicae Graph Theory, 24:5-21, 2004.

[8] M.S. Chen, K.G. Shin, D.D. Kandlur, "*Addressing, Routing and Broadcasting in Hexagonal Mesh Multiprocessors*", IEEE Trans. Computers, vol. 39, pages 10-18, 1990.

[9] M.B. Cozzens and F.S. Roberts, "*$T-$colorings of graphs and the Channel Assignment Problem*", Congr. Numer., 35, pp 191–208, 1982.

[10] M.B. Cozzens and F.S. Roberts, "*Greedy algorithms for $T-$colorings of complete graphs and the meaningfulness of conclusions about them*", J. Combin. Inform. System Sci., 16, pp. 286–299, 1991.

[11] W.Hale," *Frequency assignment theory and applications*", Proc. IEEE, 68, pp. 1497–1514, 1980.

[12] M. Kchikech, R. Khennoufa and O. Togni," *Linear and cyclic radio $k-$labelings of trees* ", Discussiones Mathematicae Graph Theory, 27 (1):105-123, 2007.

[13] R. Khennoufa and O. Togni, "The radio antipodal and radio numbers of the hypercube", Ars Combin., To appear.

[14] E. Kranakis, H. Singh, and J. Urrutia," *Campass routing in Geometric graphs*", proc. 11th Canadian Conf. Computational Geometry (CCCG-99), pages 51-54,1999.

[15] D. Liu," *Radio number for trees*", Manuscript, 2006.

[16] D. Liu and M. Xie,"*Radio Number for Square Cycles*", Congr. Numerantium, 169 : 105 - 125, 2004.

[17] Liu and X. Zhu, "Multi-level distance labelings and radio number for paths and cycles", SIAM J. Discrete Math., 3, pp. 610–621, 2005.

[18] B.H. Metzger, "*Spectrum Management Technique*", Paper presented at 38th National ORSA Meeting, Detroit, MI 1970.

[19] G. Ringel, "*Theory of Graphs and its Applications*", Proceedings of the Symposium Smolenice 1963, Prague Publ.House of Czechoslovak Academy of Science, pp.162, 1964.

[20] F. Roberts, "T -colorings of graphs: recent results and open problems", Discrete Math., 93, pp. 229–245, 1991.

[21] A. Rosa, "*Cyclic Steiner triple systems and labelings of triangular cacti*", Scientia,Vol 1, pages 87-95, 1988.

[22] R. Tosic, D. Masulovic0, I. Stojmenovic, J. Brunvoll, B.N. Cyvin, and S.J. Cyvin, "*Enumeration of polyhex Hydrocarbons up to* $h = 17$", J. Chemical Information and Computer Sciences, vol. 35, pages 181-187, 1995.

[23] K. G. Shin, HARTS, "*A Distributed Real – Time Architecture*", Computer, vol. 24, pages 25-35, 1991.

[24] P. Zhang, "*Radio labelings of cycles*", Ars combin., Vol. 65, pages 21-32, 2002.