Radio Antipodal Number of Certain Graphs

Albert William and Charles Robert Kenneth
Department of Mathematics, Loyola College, Chennai, India
charlesrobert2@yahoo.co.in

Abstract

Let $G = (V, E)$ be a graph with vertex set V and edge set E. Let $diam(G)$ denote the diameter of G and $d(u, v)$ denote the distance between the vertices u and v in G. An antipodal labeling of G with diameter d is a function f that assigns to each vertex u, a positive integer $f(u)$, such that $d(u, v) + |f(u) - f(v)| \geq d$, for all $u, v \in V$. The span of an antipodal labeling f is $max\{|f(u) - f(v)|: u, v \in V(G)\}$. The antipodal number for G, denoted by $an(G)$, is the minimum span of all antipodal labelings of G. Determining the antipodal number of a graph G is an NP-complete problem. In this paper we determine the antipodal number of certain graphs with diameter equal to 3 and 4.

Keywords: Labeling, radio antipodal numbering, diameter.

1. Introduction

Let G be a connected graph and let k be an integer, $k \geq 1$. A radio k- labeling f of G is an assignment of positive integers to the vertices of G such that $d(u, v) + |f(u) - f(v)| \geq k + 1$ for every two distinct vertices u and v of G, where $d(u, v)$ is the distance between any two vertices u and v of G. The span of such a function f, denoted by $\text{sp}(f) = max\{|f(u) - f(v)|: u, v \in V(G)\}$. Radio k- labeling was motivated by the frequency assignment problem [3]. The maximum distance among all pairs of vertices in G is the diameter of G. The radio labeling is a radio k- labeling when $k = diam(G)$. When $k = diam(G) - 1$, a radio k- labeling is called a radio antipodal labeling. In otherwords, an antipodal labeling for a graph G is a function, $f: V(G) \to \{0,1,2, \dots\}$ such that $d(u, v) + |f(u) - f(v)| \geq diam(G)$. The radio antipodal number for G, denoted by $an(G)$, is the minimum span of an antipodal labeling admitted by G. A radio labeling is a one-to –one function, while in an antipodal labeling, two vertices of distance $diam(G)$ apart may receive the same label.

The antipodal labeling for graphs was first studied by Chartrand et al.[8], in which, among other results, general bounds of $an(G)$ were obtained. Khennoufa and Togni [10] determined the exact value of $an(P_n)$ for paths P_n. The antipodal labeling for cycles C_n was studied in [4], in which lower bounds for $an(C_n)$ are obtained. In addition, the bound for the case $n \equiv 2(mod\ 4)$ was proved to be the exact value of $an(C_n)$, and the bound for the case $n \equiv 1(mod\ 4)$ was conjectured to be the exact value as well [7] . Justie Su-tzu Juan and Daphne Der-Fen Liu [9] confirmed the conjecture mentioned above. Moreover they determined the value of $an(C_n)$ for the case $n \equiv 3(mod\ 4)$ and also for the case $n \equiv 0(mod\ 4)$. They improve the known lower bound [4] and give an upper bound. They also conjectured that the upper bound is sharp.

In this paper we obtain the radio antipodal number of the n-star graph, r-regular caterpillars and r-fan graph.

2. Main Results

We begin with the following result.

Theorem 1 [4]: Let G be a connected graph of diameter $d \geq 3$. Then $an(G) \geq 2 + \Delta(d-2)$, where Δ is the maximum degree of G and d is the diameter of G.

Definition 1: A uniform n-star split graph ST_r^n contains a clique K_n such that the deletion of the nc_2 edges of K_n partitions the graph into n independant star graphs S_{r+1}. See Figure 1. The number of vertices in ST_r^n is $n(r+1)$ and the number of edges is $(nr + nc_2)$. In the following theorem, we arrive at an improved lower bound for $an(G)$, when G is ST_r^n and $r > n-3$.

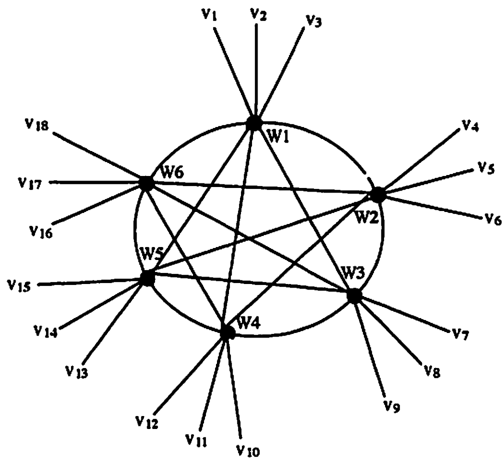

Figure 1 ST_3^6 with diameter 3

Theorem 2 Let G be an uniform n- star split graph ST_r^n. Then

$$an(G) \geq \begin{cases} \Delta + 2, if\ r \leq n-3. \\ \Delta, if\ r > n-3. \end{cases}$$

Proof. By the definition of radio-antipodal labeling, the difference between the labeling of adjacent vertices is at least two. Hence the vertices of K_n should receive labels 1, 3,...2n-1. No pendent vertex receives the label 1,3,...or 2n-1. Consider the vertex v labeled 3. The pendent vertices adjacent to v cannot receive labels 2 and 4. There are $n-3$ odd labels available to label these r pendent vertices. If $r \leq n-3$, then $an(G) \geq 2n-1 \geq \Delta$. The difference between the labels of vertices at distance two apart must be at least one. Hence, if $r > n-3$, the pendent vertices adjacent to vertex labeled 3 receive labels 6, 8,... $2n-2, 2n, 2n+1, 2n+2, \ldots 2n + (r-(n-1))$. Therefore $an(G) \geq 2n - 1 + (r-(n-3)) = n + r + 2 = \Delta + 3$.

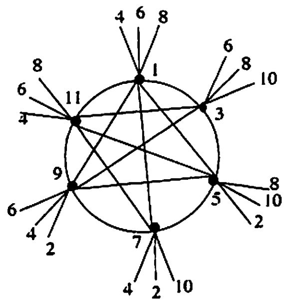

Figure 1(a) ST_3^6 **with diameter 3**

We proceed to prove that the bound obtained in Theorem 2 is sharp.

Algorithm n- star split graph ST_r^n.

- Label the vertices of C in the clockwise sense as $1, 3, \ldots (2n-1)$.
- Label the r pendent vertices adjacent to vertex labeled i on C with the first r well defined positive integers in the sequence $2, 4, \ldots i-2, i+4, i+6, \ldots 2n-2, 2n, 2n+1, \ldots$

Proof of correctness: Labels of vertices adjacent on C differ by 2. Similarly labels of pendent vertices incident at the same vertex on C differ by 2. Again two pendent vertices receiving the same label are at distance 3 apart. Then the radio antipodal number of a uniform n- star split graph ST_r^n is

$$an(G) = \begin{cases} \Delta + 2, if\ r \leq n-3. \\ \Delta, if\ r > n-3. \end{cases}$$

Definition 2: A caterpillar $C(k)$ is a tree in which the removal of vertices of degree 1 leaves a path of length k. The path is called the backbone of $C(k)$. Further if there are r pendant edges incident at every vertex of the backbone, then the caterpillar is an r -regular caterpillar and is denoted by $C(r,k)$.

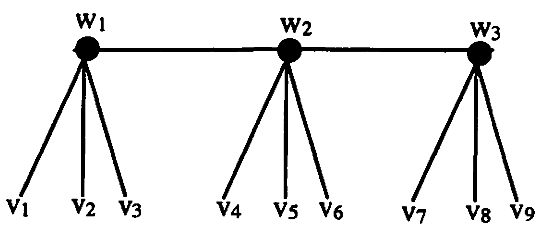

Figure 2 Caterpillar C **(3,2) with diameter 4**

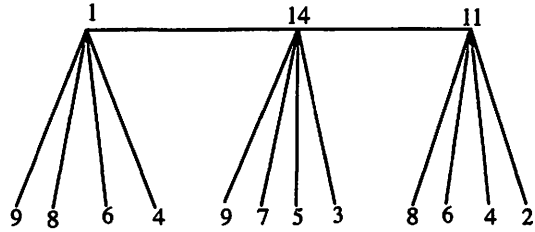

Figure 2(a) Caterpillar $C(4,2)$ with diameter 4

We give a class of caterpillars with diameter 4 for which the bound in Theorem 1 is sharp.

Theorem 3: Let G be a r -regular caterpillar of diameter 4. Then $an(G) = 2 + \Delta(d-2)$.

Proof: Label the vertices of the backbone of $C(k)$ from right to left as $2r + 3, 2r + 6, 1$. Label the r pendent edges incident at vertex labeled $2r + 3$ as $2,4,\ldots 2r$, vertex labled $2r + 6$ as $3,5,\ldots 2r+1$ and the vertex labeled 1 as $4, 6,\ldots 2r+2$. See Figure 2(a). Label v pendent vertices incident at the same vertex on the backbone differ by 2. Vertices receiving the same label are at distance 4 apart. The backbone vertices receiving labels $2r + 3$ and $2r + 6$ are adjacent, but $(2r + 6) - (2r + 3) = 3$.

Definition 3: An uniform r –fan split graph SF_r^n contains a star S_{n+1} such that the deletion of the n edges of S_{n+1} partitions the graph into n independent fans F_r and an isolated vertex called the centre of SF_r^n. See Figure 3. The number of vertices in SF_r^n is $nr + 1$ and the number of edges is $2r - 3$.

Definition 4: A fan graph, denoted by F_n, is a path P_n plus an extra vertex connected to all vertices of the path P_n.

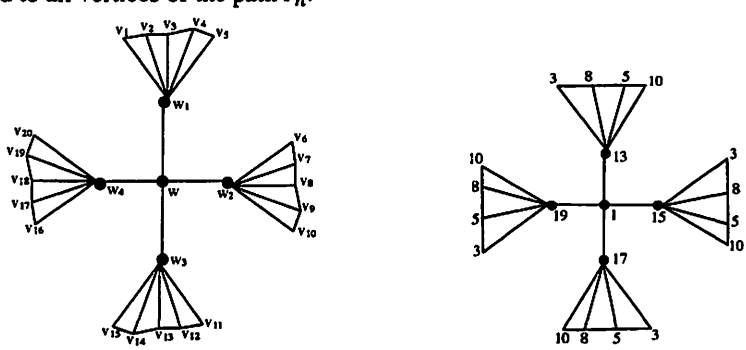

Figure 3 SF_4^5 **- fan with diameter 4**

Theorem 3: Let G be an uniform r -fan split graph SF_r^n. Then

$$an(G) \leq 2\left(\left\lfloor\frac{r}{2}\right\rfloor + n + 2\right) + 1.$$

Proof: Let w_i be the centre of the fan $F_r^i, 1 \leq i \leq n$, w be the centre of SF_r^n. Let remaining vertices of F_r^i be named as $v_{(i-1)r+1}, \ldots, v_{ir}$ and are adjacent to w_i $1 \leq i \leq n$.

Define a mapping $f: V(G) \rightarrow W$ such that $\left(v_{(2i-1)+(j-1)r}\right) = 2i + 1$, $i = 1,2, \ldots, \left\lceil\frac{r}{2}\right\rceil, 1 \leq j \leq n$.
$f\left(v_{2i+(j-1)r}\right) = 2i + 6$, i =1, 2, 3... $\left\lfloor\frac{r}{2}\right\rfloor$.
$f(w_i) = 2\left(\left\lfloor\frac{r}{2}\right\rfloor + i + 4\right)$, $i = 1,2,3 \ldots n$. $f(\mathrm{w}) = 1$. See Figure 3. By Theorem 1. This shows that $2r + 4 \leq an(G) \leq 2\left(\left\lfloor\frac{r}{2}\right\rfloor + n + 2\right) + 1$.

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