An Approximation Algorithm for the Achromatic Number of Circulant graphs $G(n;\pm\{1,2\}), G(n;\pm\{1,2,3\})$

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Abstract

The achromatic number for a graph $G = (V, E)$ is the largest integer m such that there is a partition of V into disjoint independent sets $(V_1,\ldots,V_m)$ such that for each pair of distinct sets V_i, V_j, $V_i \cup V_j$ is not an independent set in G. In this paper we present an $O(1)$-approximation algorithm to determine the achromatic number of Circulant graphs $G(n;\pm\{1,2\}), G(n;\pm\{1,2,3\})$.

Keywords: achromatic number, approximation algorithms, *NP*-completeness, Graph algorithms.

1. Introduction

A *proper* coloring of a graph $G = (V, E)$ is an assignment of colors to the vertices of G such that adjacent vertices are assigned different colors. A proper coloring of a graph G is said to be *complete* if for every pair of colors i and j there are adjacent vertices u and v colored i and j, respectively. The *achromatic number* of the graph G is the largest number m such that G has a complete coloring with m colors. Equivalently there is a partition of V into disjoint independent sets $(V_1,\ldots,V_m)$ such that for each pair of distinct sets V_i, V_j, $V_i \cup V_j$ is not an independent set in G.

The *chromatic number* problem is to minimize the number of colors over all proper colorings of a given graph. It is shown that the *achromatic number* is a maximinimal counterpart of *chromatic number*, by defining a partition-related partial order on the set of all proper colorings of G [20]. The *pseudoachromatic number* of a graph is the largest number of colors in a (not necessarily proper) vertex coloring of the graph such that every pair of distinct colors appears on the endpoints of some edge. Yegnanarayanan [29] determined pseudoachromatic number for graphs such as cycles, paths, wheels, certain complete multipartite graphs, and for other classes of graphs. Hedetniemi[12] conjectured that the two parameters *achromatic number* and *pseudoachromatic number* are equal for all trees which was disproved later [6]. Small Communication Time task systems show that the achromatic number of the co-comparability graph is an upper bound on the minimum number of processors [22].

2. An Overview of the Paper

The *achromatic number* was introduced by Harary, Hedetniemi and Prins [11,12]. The survey articles by Hughes and MacGillivray [13] and Edwards[5] contain huge collection of references of research papers related to achromatic problem.

Computing achromatic number of a general graph was proved to be *NP*-complete by Yannakakis and Gavril[27].A simple proof of this fact appears in[10]. Farber et al. [7] show that the problem is *NP*- hard on bipartite graphs. It was further proved that the achromatic number problem remains *NP*- complete even for connected graphs which are both interval graphs and cographs simultaneously [1]. Cairnie and Edwards [2], Manlove and McDiarmid [20] show that the problem is *NP*- hard even on trees. Further it is polynomially solvable for paths, cycles [5], complete bipartite graphs [19] and union of paths [21].

Since achromatic optimization problem is *NP*-hard, most of the research studies related to achromatic problem focus on approximation algorithms. An *approximation algorithm* for a problem, loosely speaking, is an algorithm that runs in polynomial time and produces an "approximate solution" to the problem. We say that an algorithm is α-approximation algorithm for a maximization problem if it always delivers a solution whose value is at least a factor $1/\alpha$ of the optimum. The parameter α is called the *approximation ratio* [8, 25].

Let n denote the number of vertices in the input graph G and let $\psi(G)$ be the achromatic number of G. It is conjectured in [3] that the achromatic number on general graphs admits an $O(\sqrt{n})$ approximation. Chaudhary and Vishwanathan [4] realize an algorithm for trees with a constant approximation ratio 7. For general graphs an algorithm that approximates the achromatic number within ratio of $O(n \,.\, \log\log n/\log n)$ is given in [18].

It is stated in [5] that "for achromatic numbers, there appear to be only a few results on special graphs apart from those for paths and cycles". Geller and Kronk[9] proved that there is almost optimal coloring for families of paths and cycles [5,13]. This result was extended to bounded trees [2]. Roichman, gives the achromatic number of hypercubes [24].

In this paper we present an $O(1)$-approximation algorithm to determine the achromatic number of circulant graphs $G(n;\pm\{1,2\})$ and $G(n;\pm\{1,2,3\})$.

3. Preliminaries

The following Lemma gives an upper bound for the achromatic number of a graph.

Lemma 1: For a graph $G = (V, E)$, if Δ is the maximum degree of a vertex in V, $\psi(G) \le \sqrt{(|V|\Delta)} + 1$.

Throughout the paper our strategy is as follows:

Identify an induced subgraph of the given graph such that a lower bound for its achromatic number is computable.

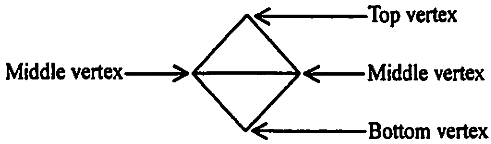


Figure 1: A Diamond-cut S

Definition 1: A *diamond-cut* is a cycle of length 4 with an additional edge joining the diagonally opposite vertices of degree 2. We denote a diamond cut by S. The top, middle and bottom vertices of S are as shown in figure 1.

We note that a diamond-cut is nothing but $K_4 - e$ where K_4 is the complete graph on 4 vertices.

Theorem 1: The achromatic number of disjoint union of t number of diamond cuts is at least $\lfloor \sqrt{1+8t} - 1 \rfloor$.

Proof: Let S be a distinct and t S denote t disjoint copies of S. Let k be an integer such that $\frac{k(k+1)}{2} < t < \frac{(k+1)(k+2)}{2}$. We first prove that when $t = \frac{k(k+1)}{2}$, $\psi(tS) \geq 2k+2$. We partition t S into $R_k \cup R_{k-1} \cup ... \cup R_1$ where each $R_i = i\,S$, $1 \leq i \leq k$.

For $0 \leq j \leq k-1$, label R_{k-j} as follows:

Label the middle vertices of alternate copies of S as $2j+1$ and $2j+2$ beginning from the first S in R_{k-j}. Label the top and bottom vertices of the left out copies of S in R_{k-j} as $2j+1$ and $2j+2$ respectively. The remaining $2(k-j)$ vertices in the S of R_{k-j} are labeled $2j+3$, $2j+4$... $2k+2$ from left to right. See Figure 2. And in R_{k-j}, vertices labeled $2j+1$ and $2j+2$ are adjacent and they in turn are adjacent to vertices labeled $2j+3$..., $2k+2$. Since none of the adjacent vertices in R_{k-i} receive the same label, the labeling is proper. Thus, the labeling induces an achromatic labeling such that the achromatic number for $\frac{k(k+1)}{2}$ S is at least $2k+2$.

Now $t < \frac{(k+1)(k+2)}{2}$ implies that

$k > \frac{-3+\sqrt{1+8t}}{2}$. Hence,

$\psi(tS) \geq 2k+2 > \sqrt{1+8t} - 1 \geq \lfloor \sqrt{1+8t} - 1 \rfloor$.

We now extend our study to the subgraphs of $G(n;\pm\{1,2,3\})$.

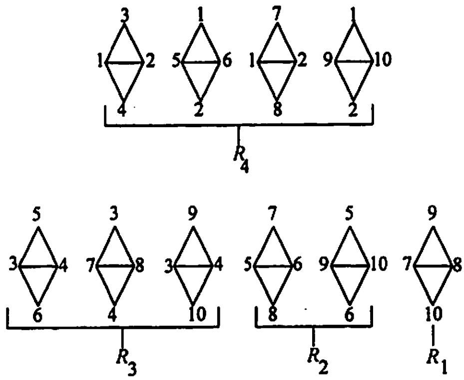


Figure 2: Achromatic labeling of 10*S*

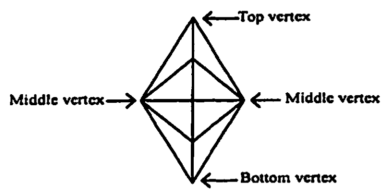


Figure 3: A *diamond-crystal*

We define the subgraph of $G(n;\pm\{1,2,3\})$ as a ***diamond-crystal***. It is denoted by *D*. See Figure 3. A disjoint union of *t* copies of *D* is denoted by *tD*.

Lemma 2:
If $r + (r-1) + (r-1) + (r-2) + (r-2) + \ldots + [r-(r-1)] + 1 \leq t$, where *t* is the number of independent *D*'s then $r \leq \sqrt{t}$.
Proof: Let *D* be a distinct diamond-crystal and *tD* denote *t* disjoint copies of *D*. We partition *tD* into $D^k \cup D^{k-1} \cup_{\ldots} \cup D^1$ where each D^k consists of $\left\lceil\frac{k-2}{4}\right\rceil + \left\lceil\frac{k-4}{4}\right\rceil + \left\lceil\frac{k-6}{4}\right\rceil + \left\lceil\frac{1}{4}\right\rceil \leq t$ where *t* is the number of copies of *D*'s.
Case (i): When $k = 4r[k \equiv 0 mod 4]$

$$r^2 \leq t$$
$$r \leq \sqrt{t}$$

Case (ii): When $k = 4r + 1[k \equiv 1 mod 4]$

$$r \leq \sqrt{1+4t} + 1 \leq \sqrt{t}$$

Case (iii): When $k = 4r + 2[k \equiv 2 mod 4]$

$$r^2 + r \leq t$$
$$r \leq \sqrt{t}$$

Case (iv): When $k = 4r + 3[k \equiv 3mod4]$

$$r^2 + 2r \leq t$$

$$r \leq \sqrt{t}$$

From all the above four cases, we conclude that if $r + (r-1) + (r-1) + (r-2) + (r-2) + \ldots + [r-(r-1)] + 1 \leq t$ then $r \leq \sqrt{t}$.

We now proceed to give an algorithm to obtain a lower bound for the achromatic number of independent D's.

Theorem 2: The achromatic number of disjoint union of t number of diamond-crystal is at least $2k+2$.

Proof: Let D be a distinct diamond-crystal and tD denote t disjoint copies of D. Let k be an integer. We prove that $\psi(tD) \leq 2k + 2$.

1. Find maximum t such that $r \leq \sqrt{t}$.
2. Partition tD into $D^k \cup D^{k-1} \cup \ldots \cup D^1$.

For $0 \leq j \leq k-1$, label D^{k-j} as follows:

Label the middle vertices of D^{k-j} as $2j+1$ and $2j+2$ beginning from the first D in D^{k-j}. The remaining $2(k-j)$ vertices in D^{k-j} are labeled $2j+3$, $2j+4 \ldots 2j+[2(k-j)+2]$ from top to bottom. See Figure 4.

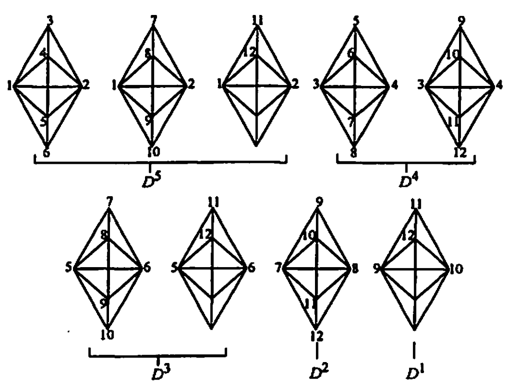


Figure 4: Achromatic labeling of 9*D*

Proof of correctness:

In D^{k-j}, vertices labeled $2j+1$ and $2j+2$ are adjacent and they in turn are adjacent to vertices labeled $2j+3\ldots, 2k+2$. Since none of the adjacent vertices in D^{k-j} receive the same label, the labeling is proper. Thus, the labeling induces an achromatic labeling such that the achromatic number is at least $2k + 2$.

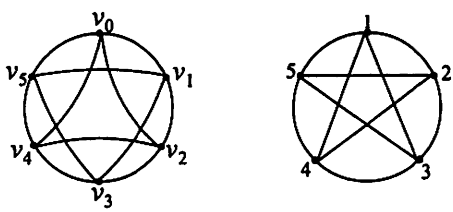
4. Circulant graphs

A graph (or diagraph) whose adjacency matrix is circulant is called a *circulant graph* (or *digraph*). Equivalently, a graph is circulant if its automorphism group contains a full-length cycle. The undirected circulant networks arise in the context of *Mesh Connected Computer* suited for parallel processing of data, such as the well-known ILLIAC type computers [27].

Definition 2: An *undirected circulant graph,* denoted by $G(n;\ \pm\{1,\ 2\ \ldots\ j\})$, $1 < j \le \left\lfloor \frac{n}{2} \right\rfloor, n \ge 3$ is defined as a graph consisting of the vertex set $V = \{0, 1 \ldots, n-1\}$ and the edge set $E = \{(i,j) : |j - i| \equiv s (\bmod n), s \in \{1, 2 \ldots j\}\}$.

Note 1: It is also clear that $G(n; \pm 1)$ is an undirected cycle.

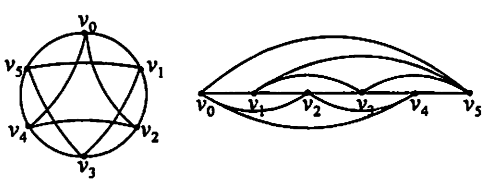
Note 2: Circulant graph $G\left(n;\pm\left\{1,2,\ldots,\left\lfloor \frac{n}{2} \right\rfloor\right\}\right)$ is a complete graph K_n and therefore the achromatic labeling of a complete graph K_n is n. See Figure 5.



(*a*) K_5 (*b*) Achromatic labeling of K_5

Figure 5: Circulant graph $G(5; \pm\{1,2\})$ is a complete graph

Achromatic number of Circulant graph $G(n;\pm\{1,2\})$



(*a*) (*b*)

Figure 6: (*a*) $G(6; \pm\{1,2\})$ (*b*) $G(6; \pm\{1,2\})$ redrawn

For our convenience, we redraw the circulant graphs $G(n;\pm\{1,2\})$ in Figure 6(*a*) as shown in Figure 6(*b*). Using the fact that diamond-cuts are induced subgraphs in circulant graphs, the following results are obtained.

Theorem 3: The circulant graph $G(n;\pm\{1,2\})$ has at least $\left\lfloor \frac{n}{6} \right\rfloor$ number of disjoint copies of $K_4 - e$ in G.

Proof: Every subgraph on 4 vertices induces a diamond-cut. Hence there are $\left\lfloor \frac{n}{6} \right\rfloor$ independent diamond-cuts in $G(n;\pm\{1,2\})$. See Figure 7.

The following theorem is straight forward as the number of edges in $G(n;\pm\{1,2\})$ is $2n$.

Theorem 4: $\psi(G(n;\pm\{1,2\})) \le \frac{1 \pm \sqrt{1+16n}}{2}$.

Theorem 5: There is an $O(1)$- approximation algorithm to determine the achromatic number of $G(n;\pm\{1,2\})$.

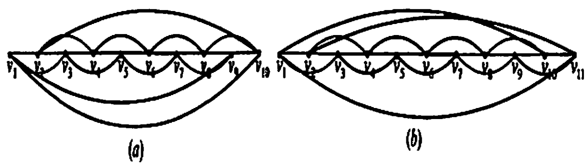


Figure 7: Induced subgraph in red color for (*a*) G(10; ±{1,2}) (*b*) G(11; ±{1,2})

Proof: The expected achromatic number for $G(n;\pm\{1,2\})$ is $\frac{1 \pm \sqrt{1+16n}}{2}$ and the lower bound realized is $\left\lfloor \sqrt{1+8\left\lfloor \frac{n}{6} \right\rfloor} - 1 \right\rfloor$ .This proves the theorem.

The following theorem is straight forward as the number of edges ε in $G(n;\pm\{1,2,3\})$ is $3n$ and $\varepsilon \ge \frac{\psi(\psi-1)}{2}$.

Theorem 6: Let $G(n;\pm\{1,2,3\})$ be circulant graphs of dimension n. Then $\psi(G(n;\pm\{1,2,3\})) \le \frac{1 \pm \sqrt{1+24n}}{2}$.

Theorem 7: There is an $O(1)$- approximation algorithm to determine the achromatic number of $G(n;\pm\{1,2,3\})$.

5. Conclusion

In this paper we present an $O(1)$-approximation algorithm to determine the achromatic number of circulant graphs $G(n;\pm\{1,2\}), G(n;\pm\{1,2,3\})$. Finding efficient approximation algorithms to determine achromatic number for toroid, Folded cubes and Augmented Cubes interconnection networks is quite challenging.

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