On Rolf Nevanlinna Prize Winners Collaboration Graph

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Abstract

The problem of determining the collaboration graph of co-authors of Paul Erdos is a challenging task. Here we take up this problem for the case of Rolf Nevanlinna Prize Winners. Even though the number of prize winners as on date is 7, the collaboration graph has 20 vertices and 41 edges and possess several interesting properties. In this paper we have obtained this graph and determined standard graph parameters for the graph as well as its complement besides probing its structural properties. Several new results were obtained.
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1 Introduction

The graphs considered in this paper are finite, simple and undirected. For any undefined terms see [1] and [9]. For any graph G, we denote by $V(G)$ and $E(G)$ the vertex set and the edge set of G respectively. The collaboration graph G has as vertices all researchers (dead or alive) from all academic disciplines with an edge joining vertices u and v if u and v have jointly published a paper or book. The distance between two vertices u and v denoted $d(u, v)$, is the number of edges in the shortest path between u and v in case if such a path exists and ∞ otherwise. Clearly $d(u, u) = 0$. We now consider the collaboration subgraph centered at Paul Erdos (1913-1996). For a researcher v, the number $d(Erdos, v)$ is called the Erdos number of v. That is, Paul Erdos himself has Erdos number 0, and his coauthors have Erdos number 1. People not having Erdos number 0 or 1 but who have published with some one with Erdos number 1 have Erdos number 2, and so on. Those who are not linked in this way to Paul Erdos have Erdos number ∞. The collection of all individuals with a finite Erdos number constitutes the Erdos component of G. 511 people have Erdos number 1, and over 5000 have Erdos number 2. In the history of scholarly publishing in Mathematics, no one has ever matched Paul Erdos's number of collaborators or papers (about 1500, almost 70% of which were joint works). Many important people in academic areas other than mathematics proper-as diverse as physics, chemistry, crystallography, economics, finance, biology, medicine, biophysics, genetics, metrology, astronomy, geology, aeronautical engineering, electrical engineering, computer Science, linguistics, psychology and philosophy do indeed have finite Erdos numbers. Also see [10] for more details.

Problem: For the sake of brevity we denote the Rolf Nevanlinna Prize Winners Collaboration Graph by G^*. In this paper we consider the problem of 1) obtaining G^*; 2) determining for G^* and its complement certain standard graph parameters; and 3) investigating the structural properties of G^*.

Construction of G^*

G^* is constructed as follows: G^* has twenty vertices and forty one edges. $V(G^*) = \{u_1, u_2, \ldots, u_{20}\}$ where u_1 = PaulErdos, u_2 = MariaMargaratKlawe, u_3 = SiemionFajtlowicz, u_4 = RobertRobinson, u_5 = George Kunthar Lorentz, u_6 = EndreSzemeredi, u_7 = LaszloLovasz, u_8 = Nathan Linial, u_9 = AlonNoga, u_{10} = BorisAronov, u_{11} = AndrejEhrenfeucht, u_{12} = MarkJerrum, u_{13} = AlokAggarwal, u_{14} = RobertEndreTarjan, u_{15} = LeslieValiant, u_{16} = A.A.Razborov, u_{17} = AviWigderson, u_{18} = PeterW.Shor, u_{19} = MadhuSudan, u_{20} = JonKleinberg. Note that the chronological order of prize winners are defined in order by u_j, $j = 14$ to 20, $E(G^*) = \{e_1, e_2, \ldots, e_{41}\}$ where $e_1 = (u_1, u_2)$, $e_2 = (u_1, u_3)$, $e_3 = (u_1, u_4)$, $e_4 = (u_1, u_5)$, $e_5 = (u_1, u_6)$, $e_6 = (u_1, u_7)$, $e_7 = (u_1, u_8)$, $e_8 = (u_1, u_9)$, $e_9 = (u_1, u_{10})$, $e_{10} = (u_2, u_8)$, $e_{11} = (u_2, u_{13})$, $e_{12} = (u_2, u_{14})$, $e_{13} = (u_2, u_{17})$, $e_{14} = (u_2, u_{18})$, $e_{15} = (u_3, u_{11})$, $e_{16} = (u_4, u_{12})$, $e_{17} = (u_5, u_{16})$, $e_{18} = (u_6, u_9)$, $e_{19} = (u_6, u_{16})$, $e_{20} = (u_6, u_{17})$, $e_{21} = (u_7, u_8)$, $e_{22} = (u_7, u_9)$, $e_{23} = (u_7, u_{17})$, $e_{24} = (u_7, u_{18})$, $e_{25} = (u_8, u_9)$, $e_{26} = (u_8, u_{13})$, $e_{27} = (u_8, u_{17})$, $e_{28} = (u_8, u_{18})$, $e_{29} = (u_9, u_{10})$, $e_{30} = (u_9, u_{17})$, $e_{31} = (u_9, u_{19})$, $e_{32} = (u_{10}, u_{13})$, $e_{33} = (u_{11}, u_{15})$, $e_{34} = (u_{12}, u_{15})$, $e_{35} = (u_{13}, u_{17})$, $e_{36} = (u_{13}, u_{18})$, $e_{37} = (u_{13}, u_{19})$, $e_{38} = (u_{13}, u_{20})$, $e_{39} = (u_{16}, u_{17})$,

$e_{40} = (u_{17}, u_{19})$, $e_{41} = (u_{19}, u_{20})$. None of the seven RNPW'S have Erdos number 1. Out of the 511 direct co-authors of Paul Erdos, with Erdos Number 1, only Nine members are connected by a path of length 1 or 2 with the RNPW'S. Out of the seven RNPW'S only five members namely $u_{14}, u_{16}, u_{17}, u_{18}, u_{19}$ have Erdos number 2, the remaining members namely u_{15}, u_{20} have Erdos number 3. G^* is shown in Figure 1.

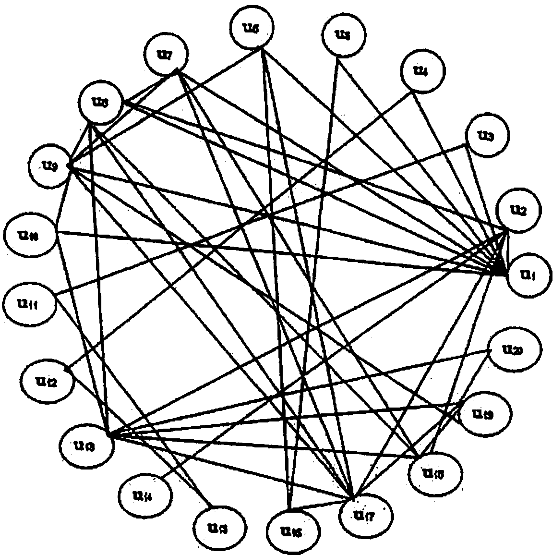
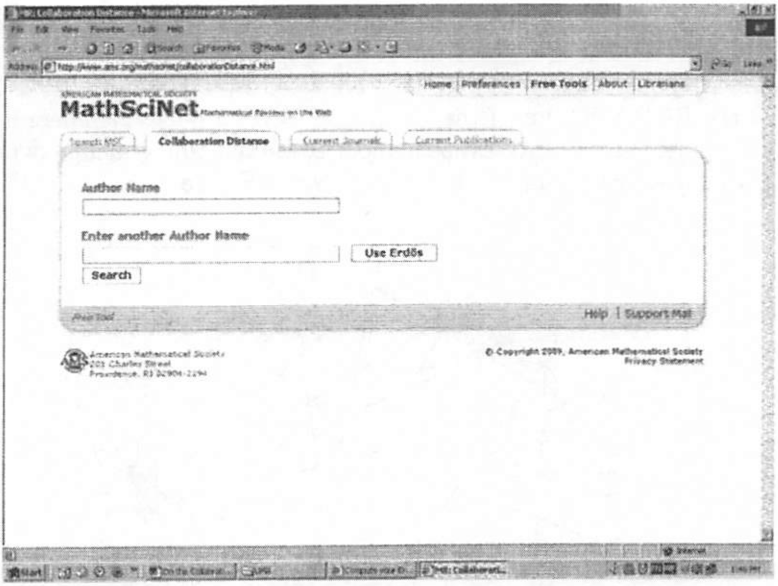


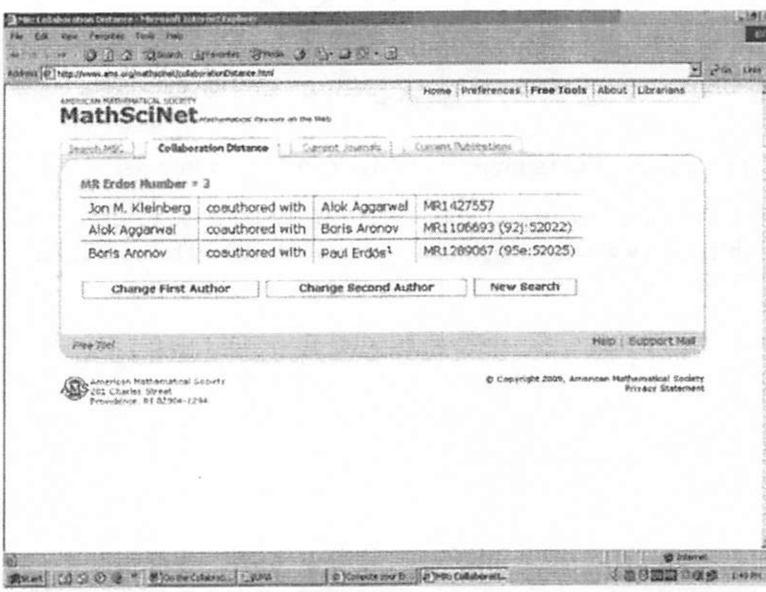
Figure 1: G^*

The method of obtaining the G^* is described as follows:

Step 1: Click on the link:
http://www.ams.org/mathscinet/collaborationDistance.html
The result of step 1 is the following screen:



Step 2: Enter the Author name and Enter another author name or click on the use Erdos icon. For example, if the author name is: Jon.M. Kleinberg and the another author name is: Paul Erdos then we obtain the following screen:



To know more details about the joint work of these authors, just click on the respective MR number.

Proceeding like this, one can obtain all the seven RNPW'S collaboration details one by one. Since the number of RNPW'S is a small number, the above procedure is recommended. It is vital to record a fact that, if there is no co author relationship at all between two persons say X and Y, then the result of our action of doing the Step 2 will be: "No path found". We have thoroughly checked all possible combinations. That is, first, we have checked the co author relationship between any of the RNPW'S with any of the 9 applicable co-authors at level 1 with Erdos number 1. This action leads to $7 \times 9 = 63$ combinations. Then we have looked for the same among 5 of the RNPW'S having Erdos Number 2. This leads to $5(5\text{-}1)/2 = 10$ combinations. Next we repeated the same for 2 of the RNPW'S at level 3 with Erdos no 3. This leads to $2 \times 5 + 2 \times 3 + 2(2-1)/2 = 17$ combinations. Also we have ascertained the coauthor relationship of the non RNPW'S at level 2 having Erdos number 2 with any of the 5 RNPW'S at the same level having Erdos number 2 and also between the non RNPW'S. This leads to $(3 \times 5) + 3(3-1)/2 = 18$ combinations. A scrupulous implementation of the above said procedure has led to the graph G^* in Figure 1.

G^* - its certain coloring parameters and their properties

Graph coloring is an important area of theoretical and practical research in combinatorics. By a coloring we mean an assignment of colors to the vertices or edges. More formally, a coloring of a graph $G(V, E)$ is a function f from $V(G)$ or $E(G)$ to the set of all natural numbers. Here we restrict our attention to only vertex colorings. Hence, the range of the coloring is only a finite subset; and if the graph is colored with k-colors, without loss of generality, we can assume the range of the coloring to the $\{1, \ldots, k\}$. A coloring of a graph G is called proper if no two adjacent vertices are assigned the same color. The minimum number of colors used in such a coloring is what is called the chromatic number of G, denoted by $\chi(G)$. A coloring (not necessarily proper) of a graph G is called a pseudocomplete coloring if for every pair of distinct colors, say, i, j there exists an edge $e = (u, v) \in E(G)$ such that u is colored i and v is colored j. The maximum number of colors used in a pseudocomplete coloring of a graph G is called the pseudoachromatic number, $\psi^*(G)$. The maximum number of colors used in a proper complete coloring of a graph G is the achromatic number, $\psi(G)$. (Note that the chromatic number of G is the minimum of colors used in a proper pseudocomplete coloring of G). Further it is easy to see that $\chi(G) \leq \psi(G) \leq \psi^*(G)$.

Proposition 1 $\chi(G^*) \leq \binom{\omega(G^*)+1}{2}$

Proof Look at G^*. As $\{u_8, u_9, u_{17}, u_{19}\}$ constitutes the complete graph on four vertices as an induced subgraph of G^*, we have $\chi(G^*) \geq 4$. Now color the vertices $u_2, u_7, u_{10}, u_{11}, u_{12}, u_{16}, u_{19}$ with color a; color the vertices u_3, u_4, u_5, u_6, u_8, u_{14}, u_{15}, u_{20} with color b; color the vertices u_1, u_{13}, u_{17} with color c; color the vertices u_9, u_{18} with d. This gives raise to a chromatic 4-coloring of G^*. This implies that $\chi(G^*) \leq 4$. Hence $\chi(G^*) = 4$. Further, it is easy to check that G^* contains no K_5, the complete graph on 5 vertices as an induced subgraph. Therefore $\omega(G^*) = 4$; As $4 = \chi(G^*) \leq 10 = \binom{4+1}{2}$, the proposition follows. □

We know that if a graph G does not contain $2K_2$ as an induced subgraph then $\chi(G) \leq \binom{\omega(G)+1}{2}$.

Proposition 2 *It is not necessary that a graph G satisfying the inequality $\chi(G) \leq \binom{\omega(G)+1}{2}$ should not contain $2K_2$ as an induced subgraph.*

Proof Clearly (u_1, u_9) and (u_{13}, u_{20}) constitutes $2K_2$ as an induced subgraph of G^*. The result now follows from Proposition 1. □

Proposition 3 *G^* contains no K_m as an induced subgraph for $5 \leq m \leq 10$.*

Proof As G^* contains no i vertices with $deg(v) \geq j$, $v \in V(G^*)$ for $7 \leq i \leq 10$ and $6 \leq j \leq 9$, K_m cannot be an induced subgraph for $7 \leq m \leq 10$. Next we observe that $\forall v \in V(G^*)$ there are 7 possible combinations namely $(u_1, u_2, u_7, u_8, u_9, u_{13})$; $(u_1, u_2, u_7, u_8, u_9, u_{17})$; $(u_1, u_2, u_7, u_9, u_{13}, u_{17})$; $(u_1, u_2, u_7, u_9, u_{13}, u_{17})$; $(u_1, u_2, u_8, u_9, u_{13}, u_{17})$; $(u_1, u_7, u_8, u_9, u_{13}, u_{17})$; $(u_2, u_7, u_8, u_9, u_{13}, u_{17})$ which can contribute a K_6 as an induced subgraph of G^*. But as $(u_2, u_{17}) \notin E(G^*)$ the combinations first to fourth and seventh and as $(u_9, u_{13}) \notin E(G^*)$, the combinations fifth and sixth all cannot give rise to a K_6 in G^*. So K_6 cannot be an induced subgraph of G^*. Finally when $m = 5$, there are $\binom{10}{4} = 252$ combinations to be examined. We leave it is an exercise for the readers to rule out each possibility and hence the proof is complete. □

Proposition 4 $\psi^*(G^*) \geq 8$.

Proof It is enough to exhibit an achromatic 8-coloring for G^*. Suppose that $\zeta(i)$ denote the i-th color class. We include the vertices of G^* appropriately into the color classes as follows. Let $\zeta(1) = \{u_{14}, u_{19}\}$, $\zeta(2) = \{u_{17}\}$, $\zeta(3) = \{u_3, u_9\}$, $\zeta(4) = \{u_8, u_{16}\}$, $\zeta(5) = \{u_5, u_{15}, u_{20}, u_{10}, u_7\}$, $\zeta(6) = \{u_1, u_{11}, u_{13}\}$, $\zeta(7) = \{u_4, u_6, u_{18}\}$, $\zeta(8) = \{u_2, u_{12}\}$. Then one can check that this coloring is both proper and pseudocomplete. So $\psi^*(G^*) \geq 8$. Finally as $\psi^*(G^*) \geq \psi(G^*)$ the proof is complete. □

Proposition 5 $9 \leq \chi(\overline{G^*}) \leq 17$, *where $\overline{G^*}$ denotes the complement of G^*.*

Proof As no two of the vertices $u_3, u_4, u_5, u_6, u_8, u_{10}, u_{14}, u_{15}$ and u_{20} are adjacent in G^*, they are all adjacent pairwise in $\overline{G^*}$ and hence K_9 will be an induced subgraph of $\overline{G^*}$. This implies that $\chi(\overline{G^*}) \geq 9$. To obtain the upper bound we appeal to the famous Nordhaus and Gaddum [14]. It says If G is a graph of order p, then 1) $|2\sqrt{p}| \leq \chi(G) + \chi(\overline{G}) \leq p+1$ and 2) $p \leq \chi(G)\chi(\overline{G}) \leq \left\lfloor (\frac{p+1}{2})^2 \right\rfloor$. □

Theorem 6 $16 \leq \psi^*(G^*) + \psi^*(\overline{G^*}) \leq 27$.

Proof Proposition 4 and Proposition 5 yields the lower bound for $\psi^*(G^*) + \psi^*(\overline{G^*})$. To see the upper bound we make use of Gupta's inequality of [8]. He proved that for any graph G of order p, $\psi(G) + \psi(\overline{G}) = 4p/3$, $\psi(G) + \psi^*(\overline{G}) = 4p/3$ and $\psi^*(G) + \psi^*(\overline{G}) = 4p/3$. We obtain the upper bound in view of this. □

G^* and its chromatic polynomial

We know that any given graph G on n vertices can be properly colored in many different ways using a sufficiently large number of colors. This property of a graph is expressed elegantly by means of a polynomial. This polynomial is called the chromatic polynomial of G and is defined as follows: The value of the chromatic polynomial $P_n(\lambda)$ of a graph with n vertices gives the number of ways of properly coloring the graph, using λ or fewer colors. Let r_i be the different ways of properly coloring G using exactly i colors. Since i colors can be chosen out of λ colors in $\binom{\lambda}{i}$ different ways, there are $\binom{\lambda}{i}$ different ways of properly coloring G using exactly i colors out of λ colors. Since i can be any positive integer from 1 to n (it is not possible to use more than n colors on n vertices), the chromatic polynomial is a sum of these terms; that is, $P_n(\lambda) = \sum_{i=1}^{n} r_i \binom{\lambda}{i}$. Clearly $r_1 = 0$, as any graph with non empty edge set requires at least two colors for properly coloring its vertices. Now Consider G^*. $r_{20} = 20!$ as G^* can be properly colored in 20! ways using 20 different colors. As $\chi(G^*) = 4$, it is easy to deduce that $r_2 = r_3 = 0$. We leave it to the readers to determine r_i for $4 \leq i \leq 19$. Hence,

Theorem 7 *The chromatic polynomial of G^* is* $P_{20}(\lambda) = \sum_{i=4}^{19} r_i \binom{\lambda}{i} + \prod_{i=0}^{19}(\lambda - i)$.

G^* and its Partitions

The Vertex-arboricity $a(G)$ of a graph G is the fewest number of subsets in a partition of the vertex set of G such that each subset induces an acyclic subgraph. Clearly $a(G) \leq \chi(G)$ for any graph G.

Proposition 8 $a(G^*) = 3$.

Proof As $a(G^*) \leq \chi(G^*) = 4$, we have $a(G^*) \leq 4$. Partition the vertex set of G^* as $V(G) = \bigcup_{i=1}^{3} V_i$ with $V_1 = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_{10}, u_{11}, u_{12}, u_{13}, u_{14}, u_{15}, u_{16}, u_{19}\}$, $V_2 = \{u_8, u_9, u_{18}, u_{20}\}$ and $V_3 = \{u_{17}\}$. Note that each V_i, $1 \leq i \leq 3$ induces an acyclic subgraph. Now it is easy to see the result. □

Proposition 9 $2 \leq \overline{a(G^*)} \leq 9$.

Proof Mitchem [12] proved that for any G of order p, 1) $\sqrt(p) \leq a(G) + a(\overline{G}) \leq \frac{p+3}{2}$; 2) $\frac{p}{4} \leq a(G)a(\overline{G}) \leq \left(\frac{(p+3)}{4}\right)^2$. In view of this we get for our G^*, $5 = a(G^*) + a(\overline{G^*}) \leq 12$ and this inturn yields that $2 \leq a(\overline{G^*}) \leq 9$. □

Observation 1 *Lick and White [11] introduced the concept of "k-degenerate graphs". A graph G is k-degenerate if $\delta(H) \leq k$ for every induced subgraph H of G. The parameter $\rho_k(G)$ of a graph is then defined as the minimum number of subsets in a partition of the vertex set of G such that each subset induces a k-degenerate graph. Clearly G^* is a 1-degenerate graph.*

G^* and connectivity properties

A set A of vertices of a graph G is a separator if $G - A$ has at least two connected components. If A induces a clique in G then we call A a clique separator. G^* has a number of clique separators. For example, $\{u_1, u_2\}$, $\{u_1, u_2, u_8\}$, $\{u_2, u_8, u_{13}, u_{18}\}$ are all clique separators of different cardinality. Further the vertices u_{14} and u_{20} are simplicial vertices, as the set of vertices adjacent to them respectively induces a clique in G^*. That is $adj(u_{14}) = \{u_2\}$, a K_1, the complete graph on one vertex and $adj(u_{20}) = \{u_{13}, u_{19}\}$, a K_2, the complete graph on two vertices, where $adj(u) = \{v : (u, v) \in E(G)\}$. It is interesting to note that the simplicial vertices need not be clique separators, as $\omega(G^*) = \omega(G^* - u_{14})$.

Proposition 10 *G^* is not a chordal graph.*

Proof We call a graph G, chordal, if every cycle in G of length at least 4 has a chord. G^* is not a chordal graph, because, the set of vertices of G^*, namely, $\{u_8, u_9, u_{10}, u_{13}\}$, even though induces a C_4, has no chord edge between the non-adjacent pair of vertices $(u_8, u_{10}), (u_9, u_{13})$. □

Proposition 11 *G^* is not self complementary.*

Proposition 12 *$\kappa(G^*) \leq \kappa'(G^*) = \delta(G^*)$, $\kappa(G^*)$, $\kappa'(G^*)$ are the vertex and the edge connectivity of G^*.*

Observation 2 *It is quite interesting to observe that a vertex disjoint clique decomposition of G^* account for only fourteen edges out of a total of forty one edges which is nearly one third of $q(G^*)$. That is $V(G) = \bigcup_{j=1}^{10} H_j$, where $H_1 = \{u_1, u_6, u_9\} \cong K_3$; $H_2 = \{u_2, u_8, u_{13}, u_{18}\} \cong K_4$; $H_3 = \{u_3, u_{11}\}$; $H_4 = \{u_4, u_{12}\}$; $H_5 = \{u_5, u_{16}\}$; $H_6 = \{u_7, u_{17}\}$; $H_7 = \{u_{19}, u_{20}\}$ all $H_i \cong K_2$, $3 \leq i \leq 7$; $H_8 = \{u_{10}\}$, $H_9 = \{u_{14}\}$, $H_{10} = \{u_{15}\}$ all $H_i \cong K_1$, $9 \leq i \leq 10$. By a clique graph $cl(G)$ of a given graph G, we mean the graph, whose vertices are the vertex-disjoint cliques of G and the edge set is constructed as follows: Introduce an edge between two clique vertices, if any vertex of one clique is adjacent to any vertex of the other clique. The clique graph $cl(G^*)$ of G^* is given in Figure 2.*

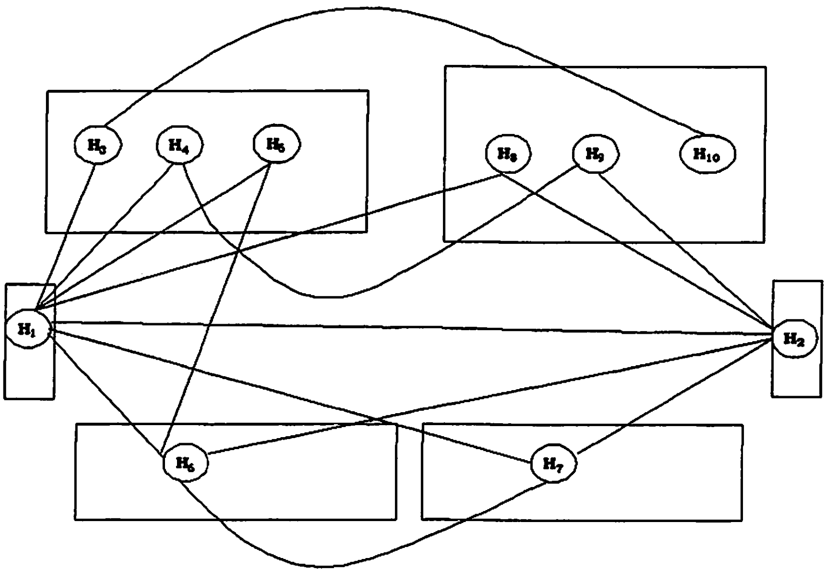


Figure 2: $cl(G^*)$

Observation 3 *We call an open walk that includes all the edges of a graph without retracing any edge a unicursal line or an open Euler line. A connected graph that has a unicursal line will be called a unicursal graph. We* *know that if a connected graph G has exactly 2k odd vertices then there exist k edge-disjoint subgraphs such that they together contain all edges of G and that each is a unicursal graph. Consider G^*. It has 8 (= 2k) odd degree vertices (with k = 4) $u_1, u_7, u_8, u_9, u_{10}, u_{13}, u_{14}, u_{16}$. Now add 4 edges to G^* between the vertex pairs (u_1, u_{14}), (u_7, u_{13}), (u_8, u_{10}), (u_9, u_{16}) to form a new graph (G^*). Since every vertex of (G^*) is of even degree, (G^*) consists of an Euler line ρ: Remove from ρ the 4 edges we just added. Then ρ will be split into 4 walks, each of which is a unicursal line. The first removal will leave a single unicursal line; the second removal will split that into two unicursal lines; and each successive removal will split a unicursal line into two unicursal lines, until there are 4 of them.*

G^*, a bounded fragmentation graph

We now proceed to check whether G^* is a bounded fragmentation graph or not? It is quite a recent interesting property introduced by Mohammad Taghi Hajiaghayi and Mahdi Hajiaghayi in [13]. We know that connectivity can be considered as a measure of the reliability of a network. Suppose that a network N is represented by an undirected graph G, in which two computers, namely nodes of the network, can communicate if and only if there is a path in G from one to other. If G is k-connected, then after removing at most $k-1$ vertices of G, the rest of G (which has $n-k+1$ vertices) is still connected. This means that if at most $k-1$ nodes of the network fail, the rest of the nodes of the network can communicate with each other. Now we define a bounded fragmentation graph. A graph G is a $(k, g(k))$-bounded fragmentation graph if $|\zeta(G[V-S])| \le |g(k)|$ for every $S \subseteq V(G)$ of size at most k, where g is a function of k. A graph G is a totally $g(k)$-bounded fragmentation graph if it is a $(k, g(k))$-bounded fragmentation graph for all $0 \le k \le n$. Here $\zeta(G)$ denote the number of components of G, where each element of $\zeta(G)$ is a connected graph. We remark that a bounded fragmentation can play a similar role in the reliability of a network like connectivity. That is, if G is a $(k, g(k))$-bounded fragmentation graph, then thereafter removing at most k vertices, we still have at least one component which has $\Omega(n)$ vertices. The reason is that after removing at most k vertices the rest of the nodes fall into at most a constant number of connected components $(g(k))$ and thus one component has at least $\Omega(n)$ vertices. Thus, after the failure of at most $k-1$ nodes of N, $\Omega(n)$ nodes in the rest of N (and not necessarily $n-k$) still can communicate with each other. So by grouping these facts, we conclude that bounded fragmentation can be considered as a generalization of connectivity. It also has another application in the reliability of a network. Suppose that we need to repair the network N temporarily by adding several links between the current nodes of the network (not by adding any new node because of its high cost) when the number of failing nodes in the networks is at most k.

If G is a $(k, g(k))$-bounded fragmentation graph, then we can simply repair the network by adding at most $g(k)-1$ number of links, which is constant. Here after removing the failure nodes, we find the connected components of G in $O(|V(G)|)$ time. Then we can connect these at most $g(k)-1$ edges among them. These two simultaneous properties of bounded fragmentation graphs cause their corresponding networks to be more reliable and robust.

Proposition 13 *G^* is a $9k$ bounded fragmentation graph.*

Proof Clearly the maximum degree of G, viz., $\Delta(G^*) = 9$, is a constant. So after removing any k vertices, $0 \leq k \leq 20$, the number of connected components is at most $g(k) = 9k$. □

Proposition 14 *G^* is totally 9-bounded fragmentation graph.*

Proof For any set $S \subseteq V(G)$ of size k, $0 \leq k \leq 20$, at least one vertex from each connected component of $G[V - S]$ is contained in any maximum independent set. Since the size of the maximum independent set is 9, we see that the number of connected components is bounded above by 9, as well. So, G is totally 9-bounded fragmentation graph. □

Proposition 15 *G^* is a totally $(k+4)$-bounded fragmentation graph.*

Proof G^* has 4 disjoint paths viz., $u_1u_2u_{14}$, $u_3u_{11}u_{15}u_{12}u_4u_{19}u_{20}$, $u_5u_{16}u_6u_9u_{10}$, $u_7u_8u_{17}u_{13}u_{18}$. Now the removal of a vertex from a path splits the path into at most two sub paths and thus at most two connected components. Thus, removing any k vertices, $0 \leq k \leq 20$, can add at most k connected components. Thus we have at most $(k+4)$-connected components. □

We say that a vertex u of G covers an edge e if u is incident with e (and conversely, e covers u). The minimum number of vertices (edges) covering all the edges (vertices) of G is called vertex- (edge) covering number of G and denoted by $\alpha_0(G)[\alpha_1(G)]$. Similarly a set A of vertices [edges] of G is said to be independent if no edge [vertex] of G is incident with more than one vertex [edge] in A. The maximum cardinality of an independent set of vertices [edges] of G is called vertex-[edge-] independence number of G and denoted by $\beta_0(G)[\beta_1(G)]$. For G^*, $\beta_0 = 9$, and the vertices are: $\{u_2, u_3, u_4, u_5, u_6, u_7, u_{10}, u_{15}, u_{19}\}$. We know that $\alpha_0 + \beta_0 = p$ where $p = |V(G)|$ and hence $\beta_0(G^*) = 9$, $p = 20$ implies $\alpha_0(G^*) = 11$, and the set of vertices which cover all the edges of G^* are $\{u_1, u_8, u_9, u_{11}, u_{12}, u_{13}, u_{14}, u_{16}, u_{17}, u_{18}, u_{20}\}$. Further we also have a result that $\alpha_1 + \beta_1 = p$ and hence we now calculate either of these parameters for G^* to find the other. Here again $\beta_1(G^*) = 9$ and the set of independent edges are $\{(u_1, u_{10}), (u_2, u_{14}), (u_3, u_{11}), (u_4, u_{12}), (u_5, u_{16}), (u_6, u_{17}), (u_7, u_9),$

(u_8, u_{13}), $(u_{19}, u_{20})\}$. So $\alpha_1(G^*) = 11$ and the set of edges which cover all the vertices of G^* are $\{(u_1, u_{10})$, (u_2, u_{14}), (u_3, u_{11}), (u_4, u_{12}), (u_5, u_{16}), (u_6, u_9), (u_7, u_8), (u_{13}, u_{17}), (u_{12}, u_{15}), (u_{13}, u_{18}), $(u_{19}, u_{20})\}$.

Proposition 16 $\beta_0(\overline{G^*}) \leq 12$.

Proof We know from [11] that if G and $\overline{G}$ are two complementary graphs of finite order p then 1) $\beta_0(G) + \beta_0(\overline{G}) \leq p + 1$ and 2) $\beta_0(G)\beta_0(\overline{G}) \leq \lfloor \frac{p+1}{2} \rfloor \lceil \frac{p+1}{2} \rceil$. In view of this we deduce that $\beta_0(\overline{G^*}) \leq 12$ as $\beta_0(\overline{G^*}) = 9$ and $p = 20$. □

Proposition 17 $7 \leq \beta_1(G^*) \leq 10$.

Proof We know from Chartrand and Schuster [4] that for a pair of complementary graphs G and $\overline{G}$ of finite order p, 1) $\lfloor \frac{p}{2} \rfloor \leq \beta_1(G) + \beta_1(\overline{G}) \leq 2\lfloor \frac{p}{2} \rfloor$ and 2) $0 \leq \beta_1(G)\beta_1(\overline{G}) \leq \lfloor \frac{p}{2} \rfloor^2$. In view of this, we have $10 \leq \beta_1(G^*) + \beta_1(\overline{G^*}) \leq 20$ and hence $1 \leq \beta_1(\overline{G^*}) \leq 11$. But the results of Cockayne and Lorimer [6] and Erdos and Schuster [7] imply, moreover, that $\left\lfloor \frac{(p+1)}{3} \right\rfloor \leq \max\{\beta_1(G^*)\beta_1(\overline{G^*})\} \leq \lfloor \frac{p}{2} \rfloor$. So, $7 \leq \max\{\beta_1(G^*)\beta_1(\overline{G^*})\} \leq 10$. □

Proposition 18 $9 \leq \alpha_1(\overline{G^*}) \leq 17$.

Proof We know from Lasker and Aucrbach [15] that if G and $\overline{G}$ are complementary graphs of order p then

1) $2\left\lfloor \frac{(p+1)}{2} \right\rfloor \leq \alpha_1(G) + \alpha_1(\overline{G}) \leq \lceil \frac{3p}{2} \rceil - 2$;

2) $\left\lfloor \frac{(p+1)}{2} \right\rfloor^2 \leq \alpha_1(G)\alpha_1(\overline{G}) \leq \left\lfloor \frac{(\lceil \frac{3p}{2} \rceil - 2)}{2} \right\rfloor \left\lceil \frac{(\lceil \frac{3p}{2} \rceil - 2)}{2} \right\rceil$;

3) $\left\lfloor \frac{(p+1)}{2} \right\rfloor \leq \min\{\alpha_1(G)\alpha_1(\overline{G})\} \leq \left\lfloor \frac{(2p+1)}{3} \right\rfloor$; In view of this we have $20 \leq \alpha_1(G^*) + \alpha_1(\overline{G^*}) \leq 28$; $100 \leq \alpha_1(G^*)\alpha_1(\overline{G^*}) \leq 196$; $10 \leq \min\{\alpha_1(G^*), \alpha_1(\overline{G^*})\} \leq 13$. Hence $9 \leq \alpha_1(G^*) \leq 17$. □

G^*, its diameter, radius, eccentricity etc.

We know that in a graph G, the distance between two vertices u and v, denoted by $d_G(v)$ is the length of the shortest path between u and v in G. The distance of a vertex v in G is defined $d_G(v) = \sum d_G(u, v)$. A vertex of minimum distance is called a median vertex of G. The median is the subgraph of G induced by its median vertices and is denoted by $M(G)$. The eccentricity of a vertex v in G denoted $e(v)$ is the number $\max_{u \in v(G)} d_G(u, v)$. The subgraph of G induced by the vertices of minimum eccentricity is the center $C(G)$ of G. The radius $r(G)$ is the minimum eccentricity of the vertices, whereas the $diam(G)$, the diameter of G is the maximum eccentricity. A vertex v is called a peripheral vertex if $e(v) = diam(G)$, and the periphery is the set of all such vertices.

Proposition 19 *$M(G^*) = K_1$, where $M(G^*)$ is the median graph of G^*.*

Proposition 20 *$C(G^*) = K_2$, where $C(G^*)$ is the center of G^*.*

Theorem 21 $M(G^*) = C(G^*)$.

Proof It follows from Proposition 19 and Proposition 20.

Corollary 21.1 $r(G^*) = 3$, where $r(G^*)$ is the radius of G^*.

Corollary 21.2 $diam(G^*) = 3$, where $diam(G^*)$ is the diameter of G^*.

Corollary 21.3 The periphery of G^* is an empty set.

Proof As G^* has no peripheral vertex the proof follows. □

Proposition 22 *G^* is the extremal graph for the inequality $r(G) \leq diam(G) \leq 2r(G)$.*

Proof It follows from Corollary 21.1 and Corollary 21.2. □

Proposition 23 $diam(\overline{G^*}) = r(\overline{G^*}) \leq 6$.

Note Nestled between the minimum eccentricity and maximum eccentricity is the average eccentricity. It was introduced by Buckley and Harary [3]. This new parameter has a practical relevance. For example, consider a communications network modeled by a graph with vertices representing the nodes of the network and edges representing the links between them. One might want to minimize the average, taken over all the nodes in the system, of the maximum time delay of a message emanating from it. This is the average eccentricity of the corresponding graph.

Theorem 24 *G^* has a supergraph H^* whose median subgraph is isomorphic to G^*.*

Deleting an edge from a graph may cause its diameter to increase or stay the same, but it cannot decrease. A graph G is diameter-minimal if for all the edges $e \in E(G)$, $diam(G - e) > diam(G)$. Any edge that can be removed from G without affecting the diameter is called superfluous. Note that diameter-minimal graphs have no superfluous edges for, let G be a diameter-minimal graph with diameter 2. Then every superfluous edge $e = (u, v)$ is contained in a triangle. Suppose not, then the removal of e would make $diam(G) \geq d(u, v) \geq 3$.

Theorem 25 *G^* can be imbedded as an induced subgraph in a diameter-minimal graph of diameter 2.*

G^* and distance degree sequence

For a vertex v in a connected graph G, let $n_i(v)$ be the number of vertices at distance i from v. The distance degree sequence of vertex v is $dd_s(v) = (n_0(v), n_1(v), \ldots, n_{e(v)}(v))$. Clearly $n_0(v) = 1$ for all v; $n_1(v) = deg(v)$. The length of the sequence $dd_s(v)$ is one more than the eccentricity of v; $\sum n_i(v) = p$. The distance degree sequence $dd_s(G)$ of a graph G consists of sequences $dd_s(v)$ of its vertices, listed in numerical order. If a particular dd_s appears k times, we list it once with k as an exponent to indicate the multiplicity. For G^*, $dd_s(u_1) = (1, 9, 8, 2)$; $dd_s(u_2) = (1, 6, 9, 4)$; $dd_s(u_1) = (1, 2, 9, 7, 1)$; etc. Similarly one can have corresponding to the distance of each vertex, a special distance sequence $sds(G)$ of a connected graph G as the list of its distance values arranged in non decreasing order. The distance values need not be consecutive integers; There need not be two vertices with maximum distance value: $sds(G)$ is derivable from $dd_s(G)$: For the sequence $dd_s(v) = (n_0(v), n_1(v), \ldots, n_{e(v)}(v))$, we have $d(v) = \sum_{i=1}^{e(v)} in_i(v)$. For instance we have for G^*, $d(u_1) = 1 \times n_1(u_1) + 2 \times n_2(u_{21}) + 3 \times n_3(u_{31}) = 31$. A graph G is geodesic if every pair of vertices u and v are joined by a unique path of length $d(u, v)$. One can see [3] for more.

Proposition 26 *G^* is not a geodesic.*

Proof We know that if every cycle of G is odd, then G is a geodesic. As G^* contains an even cycle: $u_1u_2u_8u_9u_1$, it is not geodesic. □

We know that, if G is geodesic, then every cycle of G of smallest length is odd. But the converse is not true. For example, every cycle of G^* of smallest length is 3, an odd number, but G^* is not a geodesic.

Proposition 27 *G^* must contain a cycle with a diagonal.*

Proposition 28 *G^* contains two cycles with no edges in common.*

Proposition 29 *G^* contains two cycles with no vertices in common even though it has only less than $3p - 5$ edges.*

Proof We know that a graph G with $p \geq 6$ vertices and $3p - 5$ edges contains two cycles with no vertices in common. It is not necessary that the converse of the above stated result be true. Clearly G^* has $p = 20$ vertices and less than or equal to $3p - 5$ edges and it has two vertex disjoint cycles $u_8u_9u_{10}u_{13}u_8$ and $u_6u_{16}u_{17}u_6$. □

We know that if G is connected with diameter d, then $2d-3-\left\lfloor\left(\frac{(d^2-d-4)}{p}\right)\right\rfloor \leq \left\lfloor\frac{(p^2-2q)}{p}\right\rfloor$. It is easy to check that G^* satisfies the inequality as L.H.S $= 8$ with $d = 6$ and R.H.S $= 15$ with $p = 20$, $q = 41$.

Proposition 30 *$g(G^*) \leq 2\ diam(G^*) + 1$, where $g(G^*)$ is the girth of G^*.*

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