Path Partitionable Graphs

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Abstract

The detour order of a graph G, denoted $\tau(G)$, is the order of a longest path in *G*. A partition (*A*, *B*) of *V*(*G*) such that $\tau(\langle A\rangle)\le a$ and $\tau(\langle B\rangle)\le b$ is called an *(a, b)-partition* of *G*. A graph *G* is called τ*-partitionable*, if *G* has (*a*, *b*)-partition for every pair (*a*, *b*) of positive integers such that $a+b=\tau(G)$. The well-known Path Partition Conjecture states that every graph is τ-partitionable. Motivated by the recent result of Dunbar and Frick [6] that if every 2-connected graph is τ-partitionable then every graph is τ-partitionable, we show that the Path Partition Conjecture is true for a large family of 2-connected graphs with certain ear-decompositions. Also we show that a family of 2-edge-connected graphs with certain ear-decompositions is τ-partitionable.

Keywords: Path partition; 2-connected graphs; 2-edge-connected graphs.

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1. Introduction

A longest path in a graph *G* is called a *detour* of *G*. The number of vertices in a detour of *G* is called the *detour order* of *G* and is denoted by $\tau(G)$. A partition (A,B) of *V*(*G*) such that $\tau(\langle A\rangle)\le a$ and $\tau(\langle B\rangle)\le b$ is called an *(a, b)-partition* of *G*. If *G* has an (*a*, *b*)-partition for every pair (*a*, *b*) of positive integers such that $a+b=\tau(G)$, then we say that *G* is τ*-partitionable*. The following conjecture is popularly known as Path Partition Conjecture.

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Path Partition Conjecture: Every graph is τ-partitionable.

The Path Partition Conjecture was discussed by Lavász and Mihok in 1981 in Szeged and treated in the theses [9] and [10]. Results supporting the Path Partition Conjecture and exploring its relationship with related conjectures refer [1-8].

An *n-detour colouring* of G is a colouring of the vertices of G such that no path of order greater than n is monocoloured. The nth *detour chromatic number,* denoted by χ_n, is the minimum number of colours required for an n-detour colouring of G. These chromatic numbers were introduced by Chartrand, Gellor and Hedetnimi in 1968 (see [4]).

If the Path Partition Conjecture is true, then the following conjecture of Frick and Bullock [8] is also true.

Frick-Bullock Conjecture: $\chi_n(G) \le \left\lceil \frac{\tau(G)}{n} \right\rceil$ for every graph G and for every $n \ge 1$.

Motivated by the recent result of Dunbar and Frick [6] that if every 2-connected graph is τ-partitionable then every graph is τ-partitionable, we show that the path partition conjecture is true for a large class of 2-connected graphs with certain ear-decomposition. Further we show that a family of 2-edge-connected graphs with certain ear-decomposition is τ-partitionable.

2. τ-partitionable 2-connected and 2-edge-connected graphs

In this section we show that a class of 2-connected and 2-edge-connected graphs are τ-partitionable.

Let G be a graph. A *path addition* to G of a path of length $l \ge 2$ is the graph obtained from G by attaching a path $xv_1v_2\dots v_{l-1}y$ between two vertices x and y of G with $l-1$ new vertices $v_1, v_2, \dots, v_{l-1}$. Similarly, a *cycle addition* to G of a cycle of length $l \ge 3$ at a vertex x of G is the graph obtained from G by attaching a cycle $xv_1v_2\dots v_{l-1}x$ at the vertex x of G with $l-1$ new vertices $v_1, v_2, \dots, v_{l-1}$. The path $xv_1v_2\dots v_{l-1}y$ is referred to an *attachment path* and the cycle $xv_1v_2\dots v_{l-1}x$ is referred as an *attachment cycle.*

Theorem 1. Let G be a τ-partitionable graph. Then a path or cycle addition of a path or cycle of order at least 3 to G is τ-partitionable.

Proof. Let G be a τ-partitionable graph. Let H be the graph formed by addition of a path or a cycle of order at least 3 to G. Let (a,b) be a pair of positive integers with $a+b=\tau(H)$ and $3\le a\le b$.

As G is τ-partitionable, for any pair of positive integers (a',b') with $a'+b'=\tau(G)$ there exists an (a',b')-partition (A',B') of $V(G)$. Choose (a',b') such that $3\le a'\le a$ and $3\le b'\le b$.

Suppose that H was obtained from G by the attachment of a path $P=xv_1v_2\dots v_ry$.

Case (a): $r=1$

Then $P=xv_1y$. If $x,y\notin A'$, then $(A'\cup\{v_1\},B')$ is an (a,b)-partition of $V(H)$. Without loss of generality we assume that $x\in A'$ and $y\in B'$. If x is not an end-vertex of a path of order a in $\langle A'\rangle_H$, then $(A'\cup\{v_1\},B')$ is an (a,b)-partition of $V(H)$. If x is an end-vertex of path of order a in $\langle A'\rangle_H$, then y is not an end-vertex of a path of order b in B' (otherwise, H would have a path of order $a+b+1>\tau(H)$). Therefore $(A',B'\cup\{v_1\})$ is an (a,b)-partition of $V(H)$.

Case (b): $r\ge 2$

Colour all the vertices of A' with red colour and all the vertices of B' with blue colour. Since the vertices $x,y\in V(G)$, they are coloured with either blue or red colour. Without loss of generality we may assume that $x\in A'$. Give v_r the opposite colour to y; give v_1 the colour blue, and the colour the vertices $v_2,\dots v_{r-1}$ alternatively red and blue, starting with v_2 in colour red. Note that P contains no induced monochrome subpath of order greater than 2. Further, no monochrome path in A' or B' can be extended to include any of the vertices $v_1,v_2,\dots,v_r$ of P.

Let R be the set of all red coloured vertices of $P-\{x,y\}$ and let S be the set of all blue coloured vertices of $P-\{x,y\}$. Then $(A'\cup R,B'\cup S)$ is an (a,b)-partition of $V(H)$.

Let C be a cycle addition to G. Let C be $xv_1v_2 \ldots v_{l-1}x$. Then by the above colouring by treating $x = y$ [Note that vertices x and v_1 always get different colours as well as the vertices x and v_{l-1} get different colours] we get an (a,b)-partition of $V(H)$.

A 2-connected graph G is said to have *τ-ear decomposition*, if $E(G)$ is partitioned into $G_0, P_1, P_2, \ldots P_t$ such that G_0 is a τ-partitionable subgraph of G and P_i, for $i \geq 1$, is an attachment path for the path addition to the (edge induced) graph formed by $G_0, P_1, P_2, \ldots, P_{i-1}$.

A 2-edge-connected graph is said to have *τ-closed-ear decomposition,* if $E(G)$ is partitioned into $G_0, P_1, P_2, \ldots P_t$ such that G_0 is a τ-partitionable subgraph of G, and P_i, for $i \geq 1$, is either an attachment path or an attachment cycle for the path or cycle addition to the graph formed by $G_0, P_1, P_2, \ldots, P_{i-1}$.

The following corollaries are immediate consequences of Theorem 1.

Corollary 1. Let G be a 2-connected graph with τ-ear decomposition $G_0, P_1, P_2, \ldots, P_t$ such that each P_i, for $1 \leq i \leq t$, contains at least three vertices. Then G is τ-partitionable.

Corollary 2. Let G be a 2-edge-connected graph with τ-closed-ear decomposition $G_0, P_1, P_2, \ldots, P_t$ such that each P_i, for $1 \leq i \leq t$, contains at least three vertices. Then G is τ-partitionable.

Corollary 3. Let G be a 2-connected graph with τ-ear decomposition $G_0, P_1, P_2, \ldots, P_t$ such that each P_i, for $1 \leq i \leq t$, contains at least three vertices. Then $\chi_n(G) \leq \left\lceil \frac{\tau(G)}{n} \right\rceil$, for all $n \geq 1$.

Corollary 4. Let G be a 2-edge-connected graph with τ-closed-ear decomposition $G_0, P_1, P_2, \ldots, P_t$ such that each P_i, for $1 \leq i \leq t$, contains at least three vertices. Then $\chi_n(G) \leq \left\lceil \frac{\tau(G)}{n} \right\rceil$, for all $n \geq 1$.

3. Discussion

We recall the definition of ear decomposition. An ear of a graph G is maximal path whose internal vertices have degree 2 in G. An ear decomposition of G is a decomposition $P_0, P_1, \ldots, P_k$ such that P_0 is a cycle and P_i for $i \geq 1$ is an ear of $P_0 \cup \ldots \cup P_i$. The fundamental theorem of Whitney states that a graph is 2-connected if and only it has an ear decomposition; furthermore, every cycle in a 2-connected graph is the initial cycle in some ear decomposition. The interesting part of the Corollary 1 is that it allows us to start the τ-ear decomposition from τ-partitionable subgraph rather than only with a cycle. For a graph G, maximal cliques, Hamiltonian subgraphs, claw-free subgraphs, bipartited subgraphs are always τ-partitionable. With this flexible choice of G_0 for the τ-ear decomposition, it would be more significant to know whether every 2-connected graph admits τ-ear decomposition or not. If every 2-connected graph admits τ-ear decomposition then the Path Partition Conjecture is true.

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