# One-factor Resolvability of Grid derived Networks

Bharati Rajan[1], Sonia K Thomas and Chris Monica M
Department of Mathematics, Loyola College, Chennai 600 034, India
ranikayathinkara@gmail.com

**Abstract**

Given a graph $G = (V, E)$, a set $W \subseteq V$ said to be a resolving set if for each pair of distinct vertices $u, v \in V$ there is a vertex $x$ in $W$ such that $d(u, x) \neq d(v, x)$. The resolving number of $G$ is the minimum cardinality of all resolving sets. In this paper, a condition is imposed on resolving sets and a conditional resolving parameter is studied for grid-based networks.

**Keywords:** resolving set, one-factor resolving set, augmented grid, extended grid.

## 1 Introduction

An interconnection network can be modeled by a graph in which a processor is represented by a node, and a communication channel between two nodes is represented by an edge between corresponding nodes. Various topologies for interconnection networks have been proposed in the literature. The tree, grid (especially the 2-dimensional grid $M_{n \times n}$), hypercube, $k$-ary $n$-cube, star graph, chordal rings, OTIS-Network and WK recursive grid are examples of common interconnection network topologies. The grid topology is the most dominant topology for today's regular tile-based NoCs(Networks on-Chips). It is well known that grid topology is very simple. It has low cost and consumes low power. Grid networks are highly distributed networks which use special routing technology. In wireless and mobile networks, grid networking has the obvious advantage that the software adapts dynamically to changes in the structure or *topology* of the network.

## 2 An Overview of the Paper

Let $G = (V, E)$ be a connected undirected graph. A vertex $w$ of $G$ *resolves* two vertices $u$ and $v$ of $G$ if $d(u, w) \neq d(v, w)$ where $d(x, y)$ denotes the

[1]This research is supported by The Major Research Project- No. F. 38-120/2009(SR) of the University Grants Commission, New Delhi, India.

distance between vertices $x$ and $y$ of $G$. A set $W \subseteq V$ is said to be a *resolving set* for $G$, if every pair of vertices in $G$ is resolved by some vertex in $W$. For an ordered set $W = \{w_1, w_2 \ \dots \ w_k\}$ of vertices in $G$ and a vertex $v$ of $G$, the representation of $v$ with respect to $W$ is the $k$-tuple

$$r(v/W) = (d(v, w_1), d(v, w_2) \ ... \ d(v, w_k))$$

The definition of a resolving set implies that $W$ resolves $G$ if every vertex has a distinct representation with respect to $W$. A resolving set $W$ of minimum cardinality is a *minimum resolving set* of $G$, and this cardinality is called the *resolving number* of $G$, denoted by $dim(G)$.

A basic problem in chemistry is to provide mathematical representation for a set of chemical compounds in a way that gives distinct representations to distinct compounds. The structure of a chemical compound is frequently viewed as a set of functional groups arrayed on a substructure. As described in [5], the structure is a labeled graph where the vertex and edge labels specify the atom and bond types, respectively. Thus, a graph-theoretic interpretation of this problem is to find a resolving set of the graph. There are applications of resolving sets to problems of network discovery and verification [1], pattern recognition and image processing, some of which involve the use of hierarchical data structures [12] and arise in areas like coin weighing problems [20], robot navigation [10], strategies for the Mastermind game [7], connected joins in graphs [18], geometrical routing protocols [11].

The first paper on the notion of a resolving set appeared as early as 1975 under the name '*locating set*' [19]. Slater [19] introduced this idea to determine uniquely the location of an intruder in a network. Harary and Melter [8] and Khuller et al. [10] discovered this concept independently and used the term *metric basis.* They called the resolving number as *minimum metric dimension.* This concept was rediscovered by Chartrand et al. [5] and also by Johnson [9] of the Pharmacia Company while attempting to develop a capability of large datasets of chemical graphs.

Garey and Johnson [6] showed that determining the minimum metric dimension (resolving number) of a graph is an $NP$-complete problem. It has been proved that this problem is $NP$-hard [10] for general graphs. Manuel et al. [13] have shown that the problem remains $NP$-complete for bipartite graphs. This problem has been studied for trees, multi-dimensional grids [10], Petersen graphs [2], torus networks [16], Benes networks [13], honeycomb networks [14], enhanced hypercubes [3], Illiac networks [4] and X-trees [15].

It is possible to define many resolving parameters for $G$ by combining the resolving property of $G$ with a common graph-theoretic property such as being connected, independent or acyclic. Saenpholphat et al. [17] have introduced connected resolving sets. A resolving set $W$ of $G$ is *connected* if the subgraph $G[W]$ induced by $W$ is a nontrivial connected subgraph of $G$. The minimum cardinality of a connected resolving set $W$ in a graph $G$ is called the *connected resolving number* $cr(G)$. A connected resolving set of cardinality $cr(G)$ is called a *cr-set* of $G$. Since every connected resolving set is a resolving set, $dim(G) \leq cr(G)$ for all connected graphs $G$. Thus $1 \leq dim(G) \leq cr(G) \leq n-1$, for every connected graph $G$ of order $n \geq 3$. A $cr$-set may have different graphical structures. This paper introduces a resolving parameter of $G$ when a resolving set of $G$ is a 1-factor. In other words, the graph $G[W]$ induced by a resolving set $W$ satisfies $G[W] \cong tK_2$, for some positive integer $t$. The minimum $t$ for which $G[W] \cong tK_2$ is called the *1-factor resolving number* of $G$ and is denoted by $onef(G)$. By 1-factor, we mean a 1-regular graph.

We determine the 1-factor resolving number for some grid derived architectures in the next section.

# 3 One-factor Resolving Number of Certain Grid Derived Networks

## 3.1 Grid Networks

A straight forward generalization of the linear (1-$D$) array is the grid (2-$D$) array. It is observed that the square grid has $n^2$ processors; diameter $2n-2$; vertex degree 2 or 4; almost symmetric. Grid network topology is one of the key network architectures in which devices are connected with many redundant interconnections between network nodes such as routers and switches.

The diameter of a grid is smaller than the diameter of a linear array with the same number of processors, but it is still high. A two-dimensional rectangular grid graph is an $m \times n$ graph $M(m, n)$ which is the Cartesian product of path graphs on $m$ and $n$ vertices respectively. A vertex in the $i^{th}$row and the $j^{th}$column of an $m \times n$ grid is labelled by $(i, j)$. See Figure 1.

For any positive integer $r$, an $r$-neighborhood of a vertex $v$ of a graph $G$ is defined by $N_r(v) = \{u \in V : d(u, v) = r\}$.

**Lemma 1** *Let* $G = M(m, n)$*, where* $4 \leq m < n$*. Then* $onef(G) > 1$.

**Proof.** We claim that no two column vertices inducing an edge resolves $G$.

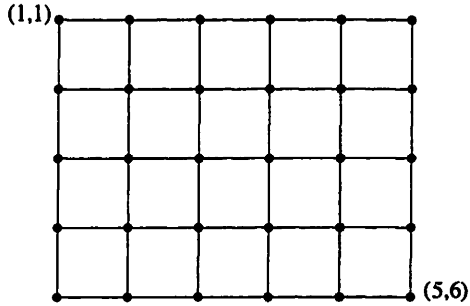


Figure 1: Grid Network $M(5,6)$

Consider the vertices $u = (i,\ j)$ and $v = (i+1, j)$. Clearly these vertices induce an edge. The vertices $(i+2, j)$ and $(i+1, j+1)$ have the same representation with respect to $u$ and $v$. This is true for $1 \leq i \leq m-2, 1 \leq j \leq n-1$. If $u = (m-1, j)$ and $v = (m, j)$, then the vertices $(m-2, j)$ and $(m-1, j+1)$ have the same representation. Similarly for $u = (i, n)$ and $v = (i+1, n)$, $1 \leq i \leq m-1$, the vertices $(i, n-1)$ and $(i-1, n)$ have the same representation with respect to $u$ and $v$. See Figure 2. The argument is same for the row vertices inducing an edge. Thus $onef(G) > 1$.

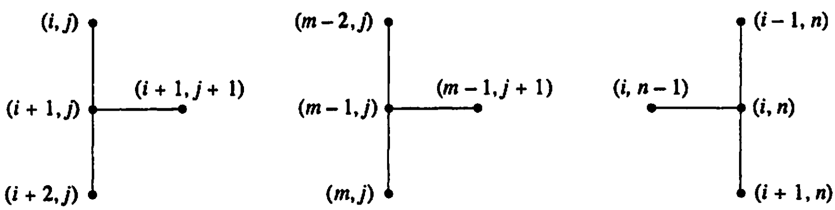


Figure 2: Proof cases of Lemma 1

**Theorem 1** *Let* $G = M(m,n)$, *where* $4 \leq m < n$. *Then* $onef(G) = 2$.

**Proof.** By Lemma 1, $onef(G) > 1$. We claim that the set

$$S = \{(1,2), (2,2), (m,2), (m,3)\}$$

resolves $G$.

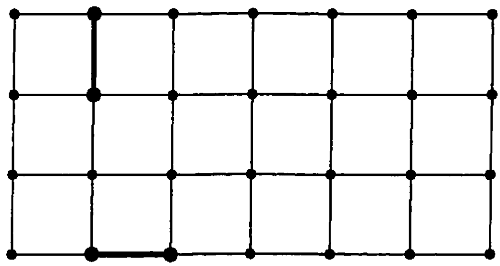

Figure 3: One-factor resolving set of $G(4,6)$

**Case 1 :** Vertices in the same row.

Consider two vertices $(i,j),(i,l)$ with $j<l$.

If $i=1$, then $d((1,j),(1,2)) \neq d((1,l),(1,2))$ except when $j=1$ and $l=3$.

Similarly, if $i=m$, then either $d((m,j),(m,2)) \neq d((m,l),(m,2))$ or $d((m,j),(m,3)) \neq d((m,l),(m,3))$. That is, if $(m,j)$ and $(m,l)$ are equidistant from $(m,2)$, then they are at unequal distances from $(m,3)$ or vice versa.

Let $1<i<m$. Then $d((i,j),(1,2)) \leq 1+d((i,l),(1,2))$, $2<j<l$.

**Case 2:** Vertices in the same column.

Consider two vertices with $(i,j),(k,j)$ with $i<k$.

If $j=2$, then $(1,2)$ resolves the vertices $(i,2),(k,2)$.

If $j=1$ or $j\geq 3$, the vertex $(m,3)$ resolves the vertices $(i,j)$ and $(k,j)$.

**Case 3 :** Vertices in different row and different column.

Consider two vertices with $(i,j),(k,l)$ with $i\neq k$, $j\neq l$.

In this case, for $i<k$ we have

$$\begin{aligned} d((i,j),(1,2)) &= d((k,l),(1,2)) \\ \text{if } i+j &= k+l \text{ and } i,j,l>1 \text{ or } i+j=k \text{ and } 1\leq j,l\leq 2 \end{aligned} \tag{1}$$

$$\begin{aligned} d((i,j),(m,2)) &= d((k,l),(m,2)) \\ \text{if } i+l &= k+j \text{ and } 1\leq j,l\leq 2 \text{ or } i+j=k \text{ and } j,l>1 \end{aligned} \tag{2}$$

Pairs of vertices $(i,j),(k,l)$, $i \neq k$, $j \neq l$ for which $i,j,k,l$ do not satisfy conditions of equation 1 are resolved by $(m,2)$. Similarly pairs of vertices $(i,j),(k,l)$, $i \neq k$, $j \neq l$ for which $i,j,k,l$ do not satisfy conditions of equation 2 are resolved by $(1,2)$.

Therefore $(1,2)$ and $(m,2)$ resolve all pairs $(i,j),(k,l)$, leaving out $(i,1),(i,3)$ for $1 \leq i \leq m$.

$d((i,1),(m,3)) \neq d((i,3),(m,3))$, by Case 1. Hence $S$ resolves all pairs of vertices in $G$. Since $S$ induces $2K_2$, the proof is complete.

**Theorem 2** *Let $G = M(m,m)$ where $m \geq 4$. Then $onef(G) = 2$.*

Proof is similar to that of Theorem 1.

## 3.2 Augmented Grid $A(m,n)$

An *augmented grid* $AM(m,n)$ is a grid $M(m,n)$ with additional edges and these additional edges are obtained by joining $(i+1,j)$ and $(i,j+1)$, $1 \leq i \leq m-1$, $1 \leq j \leq n-1$. See Figure 4.

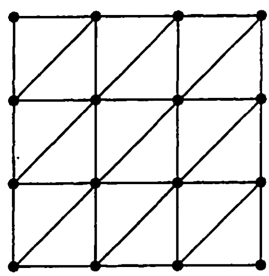

Figure 4: Augmented Grid $AM(4,4)$

**Lemma 2** *Let $G = AM(m,m)$, where $m \geq 4$. Then $onef(G) > 1$.*

As in the proof of Lemma 1, we can show that no two column or row vertices inducing an edge resolves pairs of vertices of $G$.

**Theorem 3** *$onef(AM(m,m)) = 2$ for $m \geq 4$.*

**Proof.** In view of Lemma 2, we exhibit a resolving set isomorphic to $2K_2$. Let

$$S = \{(1,2),(2,2),(m,2),(m,3)\}$$

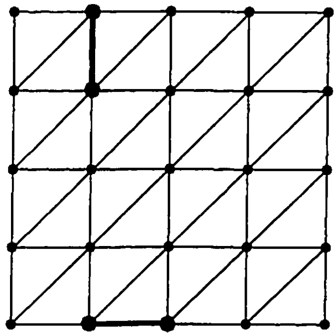

Figure 5: One-factor resolving set in $AM(5,5)$

**Case 1:** Vertices in the same row.

Let $(i,j),(i,l)$ with $j<l$ be two vertices of $AM(m,m)$. Then $d((i,j),(m,2)) \leq 1+d((i,l),(m,2))$ for $j<l$.

**Case 2:** Vertices in the same column.

Consider two vertices $(i,j),(k,j), i \neq k$. In this case either $(1,2)$ or $(2,2)$ resolves $(i,j)$, and $(k,j)$.

**Case 3:** Vertices in different row and different column.

Let $(i,j),(k,l), i \neq k, j \neq l$ be two vertices of $AM(m,n)$. Here $d((m,2),(k,l)) \leq 1+d((m,3),(i,j)), i<k, j<l$. Thus $S$ resolves pairs of vertices in $AM(m,m)$. Since $S$ is isomorphic to $2K_2$, $onef(AM(m,m)) = 2$.

Similarly, we have the following result.

**Theorem 4** *$onef(AM(m,n))=2$, where $4 \leq m < n$.*

Proof is similar to that of Theorem 2.

### 3.3 Extended Grid $EX(m,n)$

*Extended grid* $EX(m,n)$ is derived from the standard $m \times n$ grid $M(m,n)$ by making each 4-cycle into a complete graph. See Figure 6.

The vertices $(1,2),(2,2),(m,n-1)$ and $(m-1,n)$ are respectively denoted by $a,b,c,d$ in $EX(m,n)$.

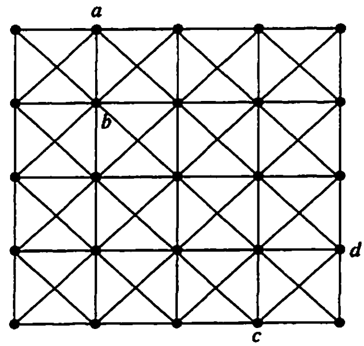


Figure 6: Extended Mesh $EX(5,5)$

**Lemma 3** *Let* $G = EX(m,m)$. *Then*

$$N_{r_1}(a) = \begin{cases} \{(1,1),(1,3),(2,1),(2,2),(2,3)\},\ r_1 = 1 \\ \\ \{(r_1+1, i+1), 0 \le i \le r_1\} \cup \\ \{(i+1, r_1+2), 0 \le i \le r_1\},\ 2 \le r_1 < m-1 \\ \\ \{(m,i), 1 \le i \le m\},\ r_1 = m-1 \end{cases}$$

$$N_{r_2}(b) = \begin{cases} \{(1,1),(1,2),(1,3),(2,1),(2,3), \\ (3,1),(3,2),(3,3)\},\ r_2 = 1 \\ \\ \{(r_2+2, i), 1 \le i \le r_2+2\} \cup \\ \{(i, r_2+2), 1 \le i < r_2+2\},\ 2 \le r_2 < m-1 \end{cases}$$

$$N_{r_3}(c) = \begin{cases} \{(m-1,m),(m-1,m-1),(m-1,m-2), \\ (m,m-2),((m,m)\},\ r_3 = 1 \\ \\ \{(m-r_3+i, m-r_3-1),(m-r_3, m-r_3+i) : 0 \le i \le r_3\}, \\ 2 \le r_3 \le m-2 \\ \\ \{(1,i) : 1 \le i \le m\}, r_3 = m-1 \end{cases}$$

$$N_{r_4}(d) = \begin{cases} \{(m-2,m-1),(m-2,m),(m-1,m-1), \\ (m,m-1),((m,m)\},\ r_4 = 1 \\ \\ \{(m-r_4+i,m-r_4),(m-r_4-1,m-r_4+i) : 0 \le i \le r_4\}, \\ 2 \le r_4 \le m-2 \\ \\ \{(i,1) : 1 \le i \le m\},\ r_4 = m-1 \end{cases}$$

In what follows we denote $N_{r_1}(x) \cap N_{r_2}(y)$ by $N_{r_1 r_2}(x,y)$

**Theorem 5** $onef(EX(m,m)) = 2$ $(m \ge 3)$.

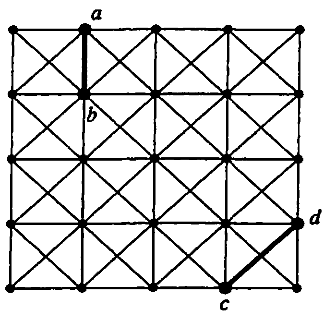


Figure 7: One-factor resolving set of $EX(5,5)$

**Proof.** Let $S = \{a,b,c,d\}$. First we define $N_{r_1 r_2}(a,b)$ and $N_{r_3 r_4}(c,d)$.

$$N_{r_1 r_2}(a,b) = \begin{cases} \{(1,1),(1,3),(2,1),(2,3)\},\ r_1 = r_2 = 1 \\ \\ \{(r_1+1,i+1) : 0 \le i \le r_1\} \\ 2 \le r_1 \le m-1,\ r_2 < r_1 \\ \\ \{(i+1,r_1+2) : 0 \le i \le r_1\} \\ 2 \le r_1 < m-1,\ r_2 = r_1 \end{cases}$$

$$N_{r_3 r_4}(c,d) = \begin{cases} \{(m-1,m-1),(m,m)\}, \ r_3 = r_4 = 1 \\ \{(m-i, m-r_3-1) : 0 \leq i \leq r_3\} \\ 1 \leq r_3 \leq m-2, \ r_3 < r_4 \\ \{(m-r_3, m-i) : 0 \leq i < r_3\} \\ 2 \leq r_3 \leq m-2, \ r_3 > r_4 \\ \{(m-r_3, m-r_3) : 2 \leq r_3 \leq m-1\}, r_3 = r_4. \end{cases}$$

Now we need to prove that for any $r_1, r_2, r_3$ and $r_4$, $| N_{r_1 r_2}(a,b) \cap N_{r_3 r_4}(c,d) | \leq 1$.

For any $r_2 < r_1$ and $r_3 < r_4$,

$$N_{r_1 r_2}(a,b) \cap N_{r_3 r_4}(c,d) = \{(r_1+1, m-r_3-1)\}.$$

Also for $r_2 = r_1$ and $r_3 > r_4$,

$$N_{r_1 r_2}(a,b) \cap N_{r_3 r_4}(c,d) = \{(m-r_1, r_1+2)\},$$

which implies that $| N_{r_1 r_2}(a,b) \cap N_{r_3 r_4}(c,d) | \leq 1$. Thus $S$ resolves pairs of vertices in $EX(m,m)$. Since $S \cong 2K_2$, $onef(EX(m,m)) = 2$. See Figure 7. Similarly we have the following result.

**Theorem 6** *Let $G = EX(m,n)$. Then $onef(G) \geq 2$ where $3 \leq m < n$.*

### 3.4 Enhanced Grid $EN(m,n)$

*Enhanced grid* $EN(m,n)$ is obtained by placing a vertex in each bounded face of an $m \times n$ grid and joining it to the corner vertices of the face.

**Theorem 7** $onef(EN(m,n)) = 2$, $3 \leq m \leq n$.

The proof is omitted.

## 4 Conclusion

In this paper, we have determined the resolving parameter, namely one-factor resolving number, for the grid based architectures. This parameter for other architectures are under investigation.

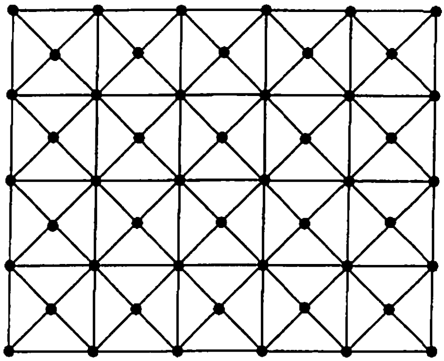

Figure 8: Enhanced Mesh $EN(5,6)$

# References


[1] Beerliova Z, Eberhard F, Erlebach T, Hall A, Hoffman M, Mihalák M, "Network Discovery and Verification", IEEE Journal on selected areas in communications, vol. 24, no. 12, pages 2168-2181, 2006.

[2] Bharati Rajan, Indra Rajasingh, Cynthia J. A., Paul Manuel, "On Minimum Metric Dimension", Proceedings of the Indonesia-Japan Conference on Combinatorial Geometry and Graph Theory, September 13-16, 2003, Bandung, Indonesia.

[3] Bharati Rajan, Indra Rajasingh, Chris Monica M, Paul Manuel, "Metric Dimension of Enhanced Hypercube Networks", The Journal of Combinatorial Mathematics and Combinatorial Computation, vol. 67, pages 5-15, 2008.

[4] Bharati Rajan, Indra Rajasingh, Venugopal P, Chris Monica M., "Minimum Metric Dimension of Illiac Networks", (accepted for publication in Ars Combintoria).

[5] Chartrand G., Eroh L., Johnson M.A., Oellermann O., "Resolvability in Graphs and the Metric Dimension of a Graph", Discrete Appl. Math., vol. 105, pages 99 - 113, 2000.

[6] Garey M. R., Johnson D. S., "Computers and Intractability: A Guide to the Theory of NP-Completeness", Freeman, New York, 1979.

[7] Goddard W., "Statistic Mastermind Revisited", J. Combin. Math. Combin. Comput., vol. 51, pages 215-220, 2004.

[8] Harary F., Melter R. A., "On the Metric Dimension of a Graph", Ars Combin., vol. 2, pages 191 - 195, 1976.

[9] Johnson M. A., "Structure-Activity Maps for Visualizing the Graph Variables Arising in Drug Design", J. Biopharm. Statist., vol. 3, pages 203-236, 1993.

[10] Khuller S., Ragavachari B., Rosenfield A., "Landmarks in Graphs", Discrete Applied Mathematics Vol. 70, no. 3, pages 217-229, 1996.

[11] Liu K, Abu-Ghazaleh N, "Virtual Coordinate Backtracking for Void Traversal in Geographic Routing", Networking and Internet Architecture, 2006.

[12] Melter R.A., Tomcscu I., "Metric Bases in Digital Geometry", Computer Vision, Graphics, and Image processing, vol. 25, pages 113-121, 1984.

[13] Paul Manuel, Mostafa I. Abd-El-Barr, Indra Rajasingh, Bharati Rajan, "An Efficient Representation of Benes Networks and its Applications", Journal of Discrete Algorithms, vol. 6, Issue 1, pages 11-19, 2008.

[14] Paul Manuel, Bharati Rajan, Indra Rajasingh, Chris Monica M., "On Minimum Metric Dimension of Honeycomb Networks, Journal of Discrete Algorithms, vol. 6, Issue 1, pages 20-27, 2008.

[15] Paul Manuel, Bharati Rajan, Indra Rajasingh, Chris Monica M., "Land marks in Binary Tree derived Architectures", (accepted for publication in Ars Combintoria).

[16] Paul Manuel, Bharati Rajan, Indra Rajasingh, Chris Monica M., "Landmarks in Torus Networks", Journal of Discrete Mathematical Sciences & Cryptography, vol. 9, no. 2, pages 263-271, 2006.

[17] Saenpholphat V., Zhang.P, "Conditional Resolvability of Graphs: A Survey", IJMMS, vol. 38, pages 1997-2017, 2003.

[18] Sebö A., Tannier E., "On Metric Generators of Graphs", Math Oper. Res.,vol. 29, no. 2, pages 383-393, 2004.

[19] Slater P.J., "Leaves of Trees", Congr. Numer., vol. 14, pages 549-559, 1975.

[20] Söderberg S., Shapiro H. S., "A Combinatory Detection Problem", Amer. Math.Monthly, vol. 70, page 1066, 1963.