# Kernel in Oriented Biregular Graphs

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**Abstract**

A kernel in a directed graph $D(V, E)$ is a set $S$ of vertices of $D$ such that no two vertices in $S$ are adjacent and for every vertex $u$ in $V \setminus S$ there is a vertex $v$ in $S$, such that $(u, v)$ is an arc of $D$. The problem of existence of a kernel is $NP$-complete for a general digraph. In this paper we solve the strong kernel problem of an oriented Biregular graphs in polynomial time.

**Keywords:** oriented graph, kernel, strong kernel number, $NP$-complete, strong orientation

# 1 Introduction

The concept of kernel is widespread and appears in diverse fields such as logic, computational complexity, artificial intelligence, graph theory, game theory, combinatorics and coding theory [3, 4]. Efficient routing among a set of mobile hosts is one of the most important functions in ad hoc wireless networks. Dominating-set-based routing to networks with unidirectional links is proposed in [1, 9]. A few years ago a new interest for these studies arose due to their applications in finite model theory. Indeed variants of kernel are the best properties to provide counter examples of $0-1$ laws in fragments of monadic second order logic [8].

A *kernel* [6] in a directed graph $D(V, E)$ is a set $S$ of vertices of $D$ such that no two vertices in $S$ are adjacent and for every vertex $u$ in $V \setminus S$ there is a vertex $v$ in $S$, such that $(u, v)$ is an arc of $D$. The minimum cardinality of all possible kernels in a directed graph $D$ is denoted by $\kappa(D)$ and is called the kernel number.The concept of kernels in digraphs was introduced in different ways [10, 15]. Von Neumann and Morgenstern [15] were the first to introduce kernels when describing winning positions in 2 person games. They proved that any directed acyclic graph has a unique kernel. Not every digraph has a kernel and if a digraph has a kernel, this kernel is not necessarily unique. All odd length directed cycles and most tournaments have no kernels [3, 4]. If $D$ is finite, the decision problem of the existence of a kernel is $NP$-complete for a general digraph [5, 14], and

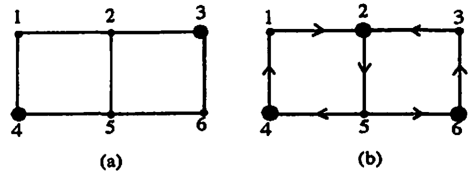


Figure 1: (a): $\gamma_i = 2$;(b): Kernel number =3

for a planar digraph with indegrees $\leq 2$, outdegrees $\leq 2$ and degrees $\leq 3$ [7]. It is further known that a finite digraph all of whose cycles have even length has a kernel [12], and that the question of the number of kernels is $NP$-complete even for this restricted class of digraphs [13].

In this paper we view the kernel problem from a different perspective. In the literature, only the existence of kernel of a digraph $G$ and its applications are extensively studied. Our aim in this paper is to investigate all strong orientations of a biregular graph $G$ and to determine the strong kernel number of $G$. This number is different from the independent domination number $\gamma_i$ for undirected graphs where $\gamma_i$ is the cardinality of a minimum independent dominating set [2]. For the graph in Figure 1 (a), $\Gamma = \{3,4\}$ is an independent dominating set. Thus $\gamma_i = 2$ where as it is easy to verify that the kernel number is 3.

An orientation of an undirected graph $G$ is an assignment of exactly one direction to each of the edges of $G$. There are $2^{|E|}$ orientations for $G$. An orientation $O$ of an undirected graph $G$ is said to be *strong* if for any two vertices $x$, $y$ of $G(O)$, there are both $(x,y)$-path and $(y,x)$-path in $G(O)$ [16].

Let $G$ be an undirected graph. Let $O_x(G)$ denote all possible orientations of a graph $G$ and $O_s(G)$ denote the set of all strong orientations of $G$. For an orientation $O \in O_x$, let $G(O)$ denote the directed graph with orientation $O$ and whose underlying graph is $G$. The kernel number of $G(O)$ is denoted by $\kappa(G(O))$.For convenience we write as $\kappa(O)$. We define the kernel number of $G$ as follows.The kernel number of $G$ is defined as $\kappa_x(G) = \min\{\kappa(O) : O \in O_x(G)\}$. Similarly we define the strong kernel number of $G$ as $\kappa_s(G) = \min\{\kappa(O) : O \in O_s(G)\}$. When there is no ambiguity we refer to $\kappa_s(G)$ as $\kappa_s$.

The *strong kernel problem* of an undirected graph $G$ is to find a kernel $K$ of $G(O)$ for some strong orientation $O$ of $G$ such that $|K| = \kappa_s$. An optimal lower bound for $\kappa_s(G)$ when $G$ is a regular graph has been obtained in [11].

# 2 Kernel in Biregular Graphs

A graph $G$ is said to be *biregular* if there exist integers $r_1$ and $r_2$ such that for every vertex $v$ in $G$, degree of $v$ is either $r_1$ or $r_2$.

Here we estimate the lower bound for the strong kernel number of certain biregular graphs. We also derive the strong kernel number for certain biregular graphs.

## 2.1 Lower bound on $\kappa_s$ for biregular graphs

The salient feature of this paper is the following result which enables us to obtain a lower bound on $\kappa_s$ for certain biregular graphs.

**Lemma 1** *Let $n, k, r_1$ and $r_2$ be integers such that $r_1 \geq 2r_2$ and $n \geq r_1 k$. Then for any $t \leq k, t + \lceil (n - r_1 t)/r_2 \rceil \geq k + \lceil (n - r_1 k)/r_2 \rceil$.*

**Proof.** If $t = k$, there is nothing to prove.

If $t < k$, then we claim that $\lceil (n - r_1 t)/r_2 \rceil - \lceil (n - r_1 k)/r_2 \rceil > k - t$. Now

$$r_1/r_2 \geq 2 \qquad (1)$$

**Case 1** ($r_2|(n - r_1 t)$ **and** $r_2|(n - r_1 k)$):

$$\begin{aligned} L.H.S &= \lceil (n - r_1 t)/r_2 \rceil - \lceil (n - r_1 k)/r_2 \rceil \\ &= (n - r_1 t)/r_2 - (n - r_1 k)/r_2 \\ &= r_1(k - t)/r_2 \\ &> k - t, \text{ by equation 1} \end{aligned}$$

**Case 2** ($r_2 \nmid (n - r_1 t)$ **and** $r_2|(n - r_1 k)$):

$$\begin{aligned} L.H.S &= \lceil (n - r_1 t)/r_2 \rceil - \lceil (n - r_1 k)/r_2 \rceil \\ &= (n - r_1 t)/r_2 + \alpha - (n - r_1 k)/r_2, 0 < \alpha < 1 \\ &= r_1(k - t)/r_2 + \alpha \\ &> k - t + \alpha, \text{ by equation 1} \\ &> k - t \end{aligned}$$

**Case 3** ($r_2|(n - r_1 t)$ **and** $r_2 \nmid (n - r_1 k)$):

Since $r_1/r_2 \geq 2$, let $r_1/r_2 = 1 + x$, $x \geq 1$.

$$\begin{aligned} L.H.S &= \lceil (n - r_1 t)/r_2 \rceil - \lceil (n - r_1 k)/r_2 \rceil \\ &= (n - r_1 t)/r_2 - [(n - r_1 k)/r_2 + \beta], 0 < \beta < 1 \\ &= r_1(k - t)/r_2 - \beta \\ &= (1 + x)(k - t) - \beta \\ &\geq (k - t) + \beta(x - 1) \\ &> k - t \end{aligned}$$

**Case 4** ($r_2 \nmid (n - r_1 t)$ **and** $r_2 \nmid (n - r_1 k)$):

$$
\begin{aligned}
L.H.S &= \lceil (n - r_1 t)/r_2 \rceil - \lceil (n - r_1 k)/r_2 \rceil \\
&= (n - r_1 t)/r_2 + \alpha - [(n - r_1 k)/r_2 + \beta], \\
&\quad 0 < \alpha < 1, 0 < \beta < 1. \\
&= r_1(k - t)/r_2 + \alpha - \beta
\end{aligned}
$$

**Subcase 4.1** ($\alpha = \beta$):

L. H. S. $= r_1(k - t)/r_2 > k - t$.

**Subcase 4.2** ($\alpha > \beta$):

Here $\alpha - \beta > 0$

$L.H.S. > r_1(k - t)/r_2$ which is $> k - t$.

**Subcase 4.3** ($\alpha < \beta$):

Here $\alpha - \beta < 0$

Therefore L. H. S. $= r_1(k - t)/r_2 - \gamma, 0 < \gamma < 1$

As before let $r_1/r_2 = 1 + x$, $x \geq 1$. Thus

$$
\begin{aligned}
L.H.S &= (1 + x)(k - t) - \gamma \\
&= (1 + x)(k - t) - \gamma \\
&\geq (k - t) + \gamma(x - 1) \\
&> k - t.
\end{aligned}
\quad \blacksquare
$$

Thus we have the following Theorem.

**Theorem 1** *Let $G$ be a graph on $n$ vertices such that each vertex is of degree either $r_1$ or $r_2$ and $r_1 \geq 2r_2$. Let $k$ be the number of vertices of degree $r_1$, subject to the condition $n \geq r_1 k$. Then $\kappa_s \geq k + \lceil (n - r_1 k)/r_2 \rceil$.*

**Proof.** For any strong orientation $O$ of $G$, every vertex has at least one indegree and at least one outdegree. Hence every vertex $v$ in $G$ of degree $r_1$ has at most $r_1 - 1$ incoming edges in $G(O)$. Suppose all the vertices of degree $r_1$ are in the kernel and dominate $k(r_1 - 1)$ vertices in $G(O)$, then the kernel members for the remaining $n - r_1 k$ vertices are to be determined. Thus $\kappa_s \geq k + \lceil (n - r_1 k)/r_2 \rceil$. ■

# 3 Strong Kernel Problem in Biregular Graphs

## 3.1 Union of two even cycles with one common vertex

It is easy to prove the following Lemma.

**Lemma 2** *Let $G$ denote the union of two even cycles with a common vertex. For $n \geq 7$, let the two cycles in $G$ on $n$ vertices be oriented in the clockwise direction. Then $G$ is strongly connected.*

**Theorem 2** *Let $G$ denote the union of two even cycles with a common vertex. Let $G$ have $n$ vertices, $n \geq 7$. Then $\kappa_s = (n - 1)/2$.*

**Proof.** Let $C_1$ and $C_2$ be the two even cycles on $2n_1$ and $2n_2$ vertices with $n = 2(n_1 + n_2) - 1$. Label the vertices of the two cycles as $1, 2, ..., 2n_1 - 1, 2n_1, 2n_1 + 1, ..., 2n_1 + 2n_2 - 1$, both in the clockwise direction starting with the common vertex $v$ as 1. See Figure 2. Then $K = \{1, 3, 5, ..., 2n_1 - 1, 2n_1 + 2, ..., 2n_1 + 2n_2 - 2\}$ is a kernel of $G$ and $|K| = n_1 + n_2 - 1$.

By Theorem 1, $\kappa_s \geq 1 + \lceil (2(n_1 + n_2) - 1 - 4)/2 \rceil$
$= 1 + \lceil (2n_1 + 2n_2 - 5)/2 \rceil$
$\geq n_1 + n_2 - 1$

Therefore $\kappa_s = n_1 + n_2 - 1$
$= \frac{n+1}{2} - 1$
$= (n-1)/2$. ■

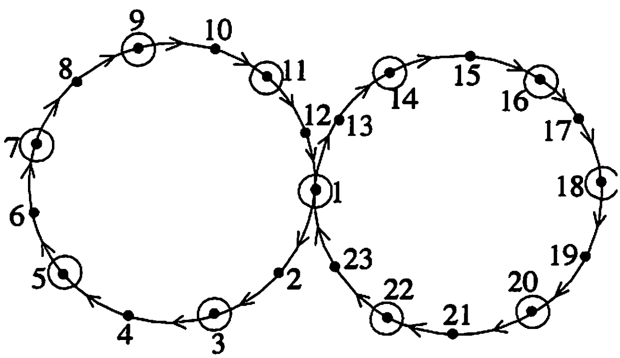


Figure 2: Encircled vertices form a kernel

## 3.2 Biregular Petersen Graphs

**Definition 1** *A generalized Petersen graph* $P(n, m), n \geq 3, 1 \leq m \leq \lfloor (n - 1)/2 \rfloor$ *consists of* $n$ *spokes* $(u_i, v_i), 1 \leq i \leq n$ *and* $n$ *inner edges* $(v_i, v_{i+m})$ *with indices taken modulo* $n$*. For convenience* $u_1, u_2, \ldots, u_n$ *are represented by* $1, 2, \ldots, n$ *and* $v_1, v_2, \ldots, v_n$ *by* $n + 1, n + 2, \ldots, 2n$ *respectively.*

Here we consider Petersen graphs with $m = 2$ and call a generalized Petersen graph $P(n, 2)$ simply a Petersen graph.

**Remark 1** *Let* $\Gamma_1$ *denote the cycle induced by the vertices* $1, 2, ..., n$*. When* $n$ *is odd, let* $\Gamma_2$ *denote the cycle induced by the vertices* $n + 1, n + 2, \ldots, 2n$*.*

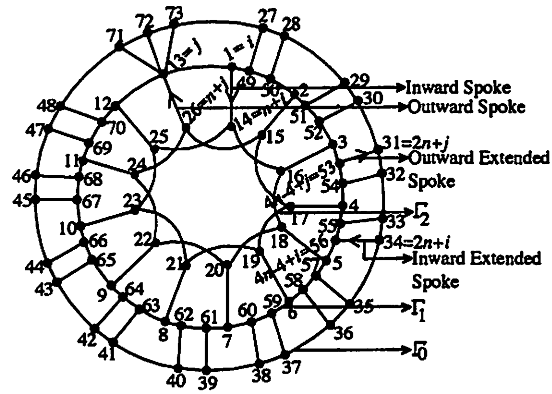


Figure 3: $BP(13,2)$

In this paper we give a variation of the definition of the Petersen graphs, namely Biregular Petersen graphs.

**Definition 2** *Let $P(n,2), n > 3$ be a Petersen graph. Consider an arc $\gamma$ of $\Gamma_1$ of length $n-2$. Subdivide each edge of $\gamma$ with 2 vertices. Let $v$ be the vertex of degree 2 in $\Gamma_1 \backslash \gamma$. Attach one pendant edge at each of the new vertices introduced in the edges of $\gamma$. These pendant edges are called extended spokes. Attach 3 pendant edges at $v$. Let $\Gamma_0$ be the cycle passing through the pendant vertices so formed. The resultant graph is called Biregular Petersen graph and it is denoted by $BP(n,2)$.*

**Remark 2** *Label the new vertices on the edge $(i, i+1)$ in $\gamma$ as $4n+(2i-5)$ and $4n+(2i-4), i = 1,2,3,...,n-2$. Label the vertices of $\Gamma_0$ beginning with the vertex adjacent to vertex labeled $4n-3$ in the clockwise direction as $2n+1, 2n+2, ..., 2n+2(n-2)$ followed by $2n+(4n-7), 2n+(4n-6)$ and $2n+(4n-5)$. See Figure 3.*

**Remark 3** *For $1 \leq i,j \leq n$, we call the oriented spoke $\overrightarrow{(i, n+i)}$ an inward spoke and the oriented spoke $\overrightarrow{(n+j, j)}$ an outward spoke. For $1 \leq i,j \leq 2n-4$, we call the oriented extended spoke $\overrightarrow{(2n+i, 4n-4+i)}$ an inward extended spoke and the oriented extended spoke $\overrightarrow{(4n-4+j, 2n+j)}$ an outward extended spoke.*

## 3.3 Orientation Algorithm

**Input:** $BP(n,2), n > 3, n$ odd

**Algorithm:**

**Step 1:** Orient the edges of cycle $\Gamma_0$ in the anticlockwise direction with an inward extended spoke and an outward extended spoke.

**Step 2:** Orient the edges of cycle $\Gamma_1$ in the clockwise direction with an inward spoke and an outward spoke.

**Step 3:** Orient the edges of $\Gamma_2$ in the clockwise direction.

**Step 4:** Orient the remaining spokes arbitrarily.

**Output:** $BP(n,2)$ is a strong orientation.

**Proof of correctness:** Let $\vec{e_1} = \overrightarrow{(i, n+i)}$ and $\vec{e_2} = \overrightarrow{(n+j, j)}$ for some $i, j, 1 \leq i, j \leq n$ be an inward spoke and an outward spoke respectively. Let $\vec{e_3} = \overrightarrow{(2n+i, 4n-4+i)}$ and $\vec{e_4} = \overrightarrow{(4n-4+j, 2n+j)}$ for some $i, j, 1 \leq i, j \leq 2n-4$ be an inward extended spoke and an outward extended spoke respectively. For $u, v \in V$, we claim that there exist directed paths from $u$ to $v$ and from $v$ to $u$. If $u, v$ lie on $\Gamma_0$, $\Gamma_1$, or $\Gamma_2$, then our claim is true since $\Gamma_0$ is oriented anticlockwise and $\Gamma_1$ and $\Gamma_2$ are oriented in the clockwise direction.

Suppose $u$ lies on $\Gamma_0$ and $v$ lies on $\Gamma_1$. The directed $(u, 2n+i)$-path on $\Gamma_0$ in the anticlockwise direction followed by $\vec{e_3}$, followed by the directed $(4n-4+i, v)$-path on $\Gamma_1$ in the clockwise direction is a path from $u$ to $v$. In the same way we trace out a directed path from $v$ to $u$. The directed $(v, 4n-4+j)$-path on $\Gamma_1$ in the clockwise direction is followed by $\vec{e_4}$, followed by the directed $(2n+j, u)$-path on $\Gamma_0$ in the anticlockwise direction is a path from $v$ to $u$. See Figure 4.

Suppose $u$ lies on $\Gamma_0$ and $v$ lies on $\Gamma_2$. The directed $(u, 2n+i)$-path on $\Gamma_0$ in the anticlockwise direction followed by $\vec{e_3}$, followed by the directed $(4n-4+i, i)$-path on $\Gamma_1$ in the clockwise direction followed by $\vec{e_1}$, followed by the directed $(n+i, v)$-path on $\Gamma_2$ in the clockwise direction is a path from $u$ to $v$. In the same way we trace out a directed path from $v$ to $u$. The directed $(v, n+j)$-path on $\Gamma_2$ in the clockwise direction followed by $\vec{e_2}$, followed by the directed $(j, 4n-4+j)$-path on $\Gamma_1$ in the clockwise direction, followed by $\vec{e_4}$, followed by the directed $(2n+j, u)$-path on $\Gamma_0$ in the anticlockwise direction is a path from $v$ to $u$. See Figure 5.

Similarly if $u$ lies on $\Gamma_1$ and $v$ lies on $\Gamma_2$, there exists directed paths from $u$ to $v$ and $v$ to $u$. Thus $G$ is strongly connected.

Based on the lower bound for the strong kernel problem, we obtain an optimal solution for Biregular Petersen graphs. This is stated in the following theorems.

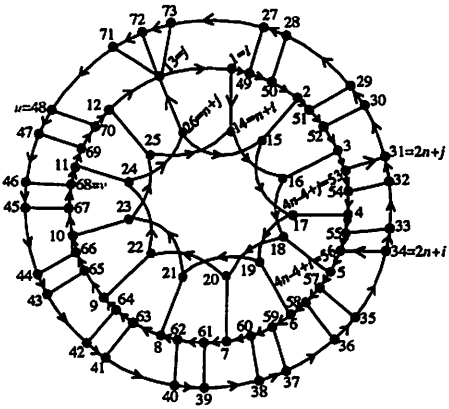


Figure 4: In $BP(13,2), u$ on $\Gamma_0$ and $v$ on $\Gamma_1$; $(\overrightarrow{1,14})$ an inward spoke, $(\overrightarrow{26,13})$ an outward spoke, $(\overrightarrow{34,56})$ an inward extended spoke and $(\overrightarrow{53,31})$ an outward extended spoke.

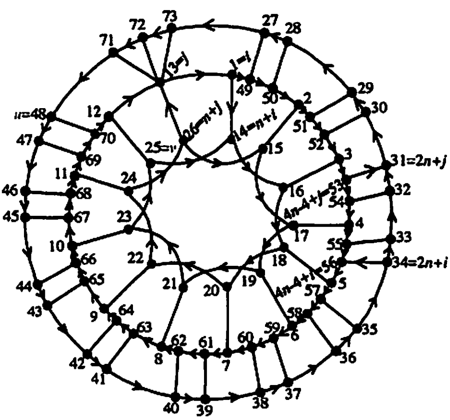


Figure 5: In $BP(13,2), u$ on $\Gamma_0$ and $v$ on $\Gamma_2$

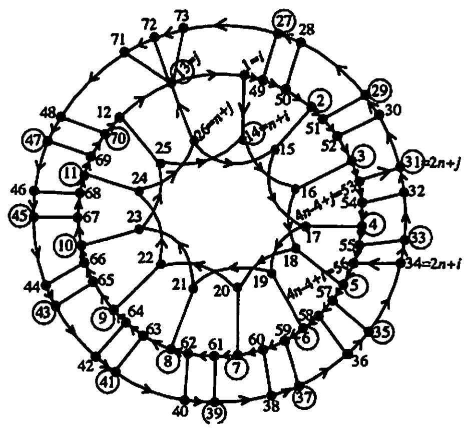

Figure 6: Encircled vertices form a kernel

**Theorem 3** *Let $G$ be $BP(n,2), n > 3$. Then* $\kappa_s = 1 + \lceil \frac{6n-11}{3} \rceil$.

**Proof.** Consider $G(O)$ with $O$ defined by the orientation algorithm. See Figure 6. It is easy to check that $K = \{2, 3, 4, ..., n-2, n, n+1, 2n+1, 2n+3, ..., 2n+(2n-5), 2n+(4n-8)\}$ form a kernel of $G$. ■

**Theorem 4** *Let $G$ be The strong kernel problem for $BP(n,2), n > 3$ is polynomially solvable.*

# 4 Conclusion

In this paper, we have determined the lower bound for the strong kernel number for biregular graphs and also proved that the strong kernel problem is polynomially solvable for union of two even cycles with one common vertex and Biregular Petersen graphs. It would be interesting to identify more biregular graphs for which the lower bound for $\kappa_s$ is attained.

# References


[1] K. M. Alzoubi, P. J. Wan and O. Frieder, *New Distributed Algorithm for Connected Dominating Set in Wireless Ad Hoc Networks*", Proc. 35th Hawaii Int. Conf. System Sciences, pages 01-07, 2002.

[2] S. Armugam and S. Velammal, *Maximum size of a connected graph with given domination parameters*, Ars Combin., 52 pages 221-227, 1992.

[3] J. Bang-Jensen and G. Gutin, *Digraphs: Theory, Algorthims and Applications*, Springer, London, 2000.

[4] C. Berge and P. Duchet, *Recent problems and results about kernels in directed graphs*, Discrete Mathematics 86, pages.27-31, 1990.

[5] V. Chvàtal, *On the computational complexity of finding a kernel*, Report No. CRM-300, Centre de Recherches Mathematiques, Universite de Montreal, 1973.

[6] Claude Berge, *Graphs,* volume 6, North Holland Publishing Co., Amsterdam, 1985.

[7] A. S. Fraenkel, *Planar kernel and Grundy with* $d \leq 3, d^+ \leq 2, d^- \leq 2$ *are* $NP$*-complete*, Discrete Applied Mathematics 3, pages.257-262, 1981.

[8] Jean-Marie Le Bars, *Counterexample of the* $0-1$ *Law for fragments of existential second-order logic: an overview*, The Bulletin of Symbolic Logic, 6(1), pages 67-82, 2000.

[9] Jie Wu, *Extended Dominating-Set-Based Routing in Ad Hoc Wireless Networks with Unidirectional Links*, IEEE Transactions on Parallel and Distributed Systems, Vol. 13, No. 9,pages 866-881, 2002.

[10] M. Kswásnik, *The generalisation of Richardson theorem*, Discussiones Math. IV, pages 11–14, 1981.

[11] Paul Manuel, Indra Rajasingh, Bharati Rajan, Joice Punitha, *Kernel in Oriented Circulant Graphs,* Combinatorial Algorithms, LNCS, Springer Berlin / Heidelberg, Vol. 5874/2009.

[12] M. Richardson, *Solutions of irreflexive relations*, Ann. of Math. 58, pages 573-590, 1953.

[13] J. L. Szwarcfiter and G. Chaty, *Enumerating the kernels of a directed graph with no odd circuits*, Inform. Process. Lett. 51, pages 149-153, 1994.

[14] J. Van Leeuwen, *Having a Grundy-numbering is NP-complete*, Report No. 207, Computer Science Dept., Pennsylvania State University, University Park, PA, 1976.

[15] J. Von Neumann and O. Morgenstern, *Theory of Games and Economic Behaviour,* Princeton University Press, Princeton, 1944.

[16] J. Xu, *Topological Structure and Analysis of Interconnection Networks*, Kluwer Academic Publishers, 2001.