On Self Orthogonal Diagonal Latin Squares

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Abstract. This paper considers latin squares of order n having $0, 1, \dots, n-1$ down the main diagonal and in which the back diagonal is a permutation of these symbols (*diagonal* squares). It is an open question whether or not such a square which is self orthogonal (i.e., orthogonal to its transpose) exists for order 10.

We consider two possible constraints on the general concept: self conjugate squares and strongly symmetric squares. We show that relative to each of these constraints, a corresponding self orthogonal diagonal latin square of order 10 does not exist. However it is easy to construct self orthogonal diagonal latin squares of orders 8 and 12 which satisfy each of the constraints respectively.

1. Introduction.

This paper reports on the results of our efforts to find a particular kind of pair of mutually orthogonal latin squares. We begin with some preliminary definitions. (For a more general discussion of latin squares, the reader should consult [2].) A *latin square of order* n is an $n \times n$ array such that each row and each column is a permutation of $N = \{0, 1, \dots, n-1\}$. Two such squares A and B are said to be *orthogonal* if for each $i, j \in N \times N$, there are k, l such that $A(k, l) = i$ and $B(k, l) = j$. A *transversal* in a latin square is a set of positions, one from each row and one from each column, which between them contain each of the symbols from N exactly once. A *transversal square* is a latin square with a transversal. For our purposes we can assume that the main diagonal of any transversal square is $(0, 1, \dots, n-1)$.

A latin square of order n is called *diagonal* if it is a transversal square and its back diagonal is a permutation of $\{0, 1, \dots, n-1\}$. For any latin square A, we denote the transpose of A by A^T and we say that A is *self orthogonal* (see [3]) if A, A^T is an orthogonal pair. Any self orthogonal latin square is necessarily a transversal square. The existence of a self orthogonal, diagonal square of order 10 is an open problem, although as shown in [3], self orthogonal squares of order 10 are known to exist. In fact, they exist for all orders except 2, 3 and 6 (see [1]).

In this paper we provide some negative evidence for the existence of a self orthogonal diagonal square of order 10. Our results are based on a combination of theoretical conclusions and computer searches. We note that our conclusions result by placing certain constraints on the general concept of a self orthogonal diagonal square. Whereas the general question of the existence of such a square of order 10 is computationally infeasible, our constraints make the search for a corresponding square tractable (or theoretically solvable). Moreover, the concepts introduced via the constraints appear to be of some independent interest in their own right.

Section 2 considers a special type of self orthogonal diagonal square, "self conjugate squares", and shows that no such square of order 10 exists. In section 3, we focus on "strongly symmetric", self orthogonal diagonal squares and again show that no such square of order 10 exists. Finally, section 4 contains concluding observations and poses some follow up questions.

2. Self conjugate latin squares.

Given an orthogonal pair A, B of latin squares of order n, we define the *conjugate pair* A^*, B^* as follows:

for $i,j \in N$,
$A^*(A(i,j), B(i,j)) = i$ and
$B^*(A(i,j), B(i,j)) = j$

A^*, B^* is again an orthogonal pair and its conjugate pair is A, B. Thus forming the conjugate pair is an idempotent operation. If A is self orthogonal, we define the *conjugate* of A to be A^* where $A^*, (A^T)^*$ is the conjugate pair of A, A^T. We call A *self conjugate* if $A = A^*$.

Up to transpose, there is only one self orthogonal diagonal square of order 4 and it is self conjugate:

0 3 1 2
2 1 3 0
3 0 2 1
1 2 0 3

There are numerous self conjugate, self orthogonal diagonal latin squares for each of the orders 8 and 12 (we give an example of such a square of order 12 in a later section). However the following theorem rules out the existence of such a square in the case of order 10.

Theorem. *If there is a self conjugate, self orthogonal diagonal latin square of order n, then*

$$n^2 - n = 0 \pmod 4.$$

Proof: Let S be the set of all pairs $\{i,j\}$ with $i,j \in N$ and $i < j$. Suppose that A is a self orthogonal diagonal square. Then the function f defined on S by $f(\{i,j\}) = \{A(i,j), A(j,i)\}$ is a bijection with no fixed point. Since A is self conjugate, it follows that f is idempotent. In order for f to be idempotent and have no fixed point, S must have an even number of elements. Thus $n(n-1)/2$ is even or $n^2 - n = 0 \pmod 4$. ∎

We note that the converse of this theorem is false — there is no self conjugate, self orthogonal diagonal latin square of order 5.

3. Strongly symmetric latin squares.

We call a self orthogonal diagonal latin square *strongly symmetric* if

$$A(i,j) + A(n-1-i, n-1-j) = n-1 \text{ for } i,j \in N.$$

As determined by a computer search, all self orthogonal diagonal squares of orders 4, 5, and 7 are strongly symmetric. Once again, for each of the orders 8 and 12, numerous such squares also exist. We were able to show by an exhaustive computer search that no such square of order 10 exists. In the remainder of this section we outline how we reduced this computation down to quite manageable size (in the case of order 10).

Let p be a bijection on N. For any latin square A, we define the square $p(A)$ by

$$p(A)(p(i), p(j)) = p(A(i,j)) \text{ for } i,j \in N.$$

If A is self orthogonal and diagonal, then $p(A)$ is self orthogonal, but not necessarily diagonal. To ensure that $p(A)$ is again diagonal, it is sufficient (but not necessary) to constrain p so that each position on the back diagonal is again mapped to a position on the back diagonal. This is equivalent to requiring that the following condition holds:

$$p(i) + p(n-1-i) = n-1 \text{ for } i \in N.$$

We will call the isomorphism p *strongly symmetric* if it satisfies this condition. For each latin square A, we let d^A be the bijection determined by A's back diagonal, that is

$$d^A(i) = A(i, n-1-i) \text{ for } i \in N.$$

If A is strongly symmetric, then d^A is also strongly symmetric. For $x \in N$, we let x' denote $n-1-x$. In the case where A has order 10, d^A, as a product of disjoint cycles, has one of the following 6 forms:

$$(uvwxy)(u'v'w'x'y')$$
$$(uvw)(u'v'w')(xyx'y')$$
$$(uvw)(u'v'w')(xy)(x'y')$$
$$(uvwxyu'v'w'x'y')$$
$$(uvwu'v'w')(xy)(x'y')$$
$$(uvwu'v'w')(xyx'y')$$

where the indicated values are all in the range $0, \ldots, 9$ and are all distinct. We shall refer to each of these six forms as a *cycle structure*.

Theorem. *Let A be a strongly symmetric, self orthogonal diagonal latin square of order 10 and let d be any permutation with the same cycle structure as d^A. Then* *there is a strongly symmetric isomorphism mapping A to a square having d as its back diagonal.*

Proof: The isomorphism p is defined as follows. We associate (arbitrarily) each cycle of d^A with a cycle of d of the same length. Now let $(a_1, \ldots, a_k)$ and $(b_1, \ldots, b_k)$ be corresponding cycles of d^A and d respectively. We see from the possible cycle structures noted above that we can consistently set $p(a_i) = b_i$ and $p(a_i') = b_i'$. Furthermore, p can be extended in this way until it is defined on all elements of d^A's cycles (the entire domain). As constructed, p is strongly symmetric. Finally, note that an entry on $p(A)$'s back diagonal, say a_{i+1} at position a_i, a_i', is mapped via p to b_{i+1} at position b_i, b_i'. Consequently $p(A)$'s back diagonal determines the permutation d and the result follows.

As a consequence of the above theorem, in searching for a strongly symmetric, self orthogonal diagonal latin square of order 10, we need only look for one with a back diagonal of each of the following six possibilities:

(0 1 2 3 4)(9 8 7 6 5)
(0 1 2)(9 8 7)(3 4 6 5)
(0 1 2)(9 8 7)(3 4)(6 5)
(0 1 2 3 4 9 8 7 6 5)
(0 1 2 9 8 7)(3 4)(6 5)
(0 1 2 9 8 7)(3 4 6 5)

As noted previously, the resulting exhaustive computer search was quite manageable and produced no latin squares. Thus we are able to conclude that there is no strongly symmetric, self orthogonal diagonal latin square of order 10.

4. Conclusions.

Suppose that A is self orthogonal diagonal square of order n. It is easy to check that an isomorphic image, $p(A)$, is diagonal if and only if the values $A(p^{-1}(i), p^{-1}(n-1-i))$ are distinct for $i \in N$. We will say two such squares are *equivalent* if they can be transformed to each other, or one to a transpose of the other, by such an isomorphism. In general, $p(A)$ need not be strongly symmetric if A is, but $p(A)$ will be self conjugate if A is. For example, suppose A is as follows:

0 5 6 2 3 7 4 1
4 1 3 7 6 2 0 5
5 6 2 0 1 3 7 4
7 4 1 3 2 0 5 6
1 2 7 5 4 6 3 0
3 0 4 6 7 5 1 2
2 7 5 1 0 4 6 3
6 3 0 4 5 1 2 7

Then A is strongly symmetric, but if p is defined by $(0\ 4\ 3\ 7)(1\ 5\ 2\ 6)$, then $p(A)$ is not strongly symmetric.

Self orthogonal diagonal latin squares which are neither self conjugate nor strongly symmetric are quite difficult to find except by transforming strongly symmetric squares as in the last example. If all such squares were transforms of strongly symmetric squares, then we would have established the non-existence of a self orthogonal diagonal square of order 10. However this is not the case. An exhaustive (computer) computation shows that the following square of order 9 is not equivalent to a strongly symmetric square.

0	5	4	6	7	2	8	3	1
2	1	0	7	5	8	4	6	3
6	3	2	0	1	4	5	8	7
5	8	1	3	6	0	7	2	4
3	7	8	1	4	6	2	5	0
1	6	7	8	3	5	0	4	2
4	0	3	2	8	7	6	1	5
8	2	5	4	0	1	3	7	6
7	4	6	5	2	3	1	0	8

There are 64 self orthogonal diagonal latin squares of order 7, in 4 equivalence classes. There are 392 such squares of order 8 and 288 self conjugate squares of order 8 (12 which are both strongly symmetric and self conjugate) in just 2 equivalence classes. One of these classes contains all the self conjugate squares. There are many self orthogonal diagonal squares of order 12 which are both strongly symmetric and self conjugate. The constraints make them easy to find by computer search. We list one below.

0	3	7	8	10	2	4	11	1	5	9	6
2	1	10	11	3	9	5	0	4	6	7	8
6	9	2	1	7	4	3	5	11	8	0	10
11	7	0	3	2	1	8	6	9	10	5	4
8	5	9	6	4	11	1	10	2	0	3	7
10	0	8	4	9	5	11	3	6	7	1	2
9	10	4	5	8	0	6	2	7	3	11	1
4	8	11	9	1	10	0	7	5	2	6	3
7	6	1	2	5	3	10	9	8	11	4	0
1	11	3	0	6	8	7	4	10	9	2	5
3	4	5	7	11	6	2	8	0	1	10	9
5	2	6	10	0	7	9	1	3	4	8	11

Although the original question of the existence of a self orthogonal diagonal latin square of order 10 remains, we have provided some negative evidence in this regard. Finally, we conjecture that if n is odd, there is no self conjugate, self orthogonal diagonal square of order n.

References

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3. A. Hedayat, *Self orthogonal latin square designs and their importance*, Biometrics **29** (1973), 393–396.