# Similar Graphs: Characterization and Subclasses

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**Abstract.** In this paper we introduce the concept of similar graphs. Similar graphs arise in the design of fault tolerant networks and in load balancing of the networks in case of node failures. Similar graphs model networks that not only remain connected, but also allow a job to be shifted to other processors without re-executing the entire job. This dynamic load balancing capability ensures minimal interruption to the network in case of single or multiple node failures and increases overall efficiency.

We define a graph to be $(m, n)$-similar if each vertex is contained in a set of at least $m$-vertices, each pair of which share at least $n$-neighbors. Several well known classes of $(2,2)$-similar graphs are characterized, for example, triangulated, comparability and co-comparability. The problem of finding a minimum augmentation to obtain a $(2,2)$-similar graph is shown to be NP-Complete.

A graph is called strongly $m$-similar if each vertex is contained in a set of at least $m$ vertices with the property that they all share the same neighbors. The class of strongly $m$-similar graphs is completely characterized.

## 1. Introduction.

In the design of networks, for example, communication networks [13], parallel architectures [9] and switching systems [3, 11], the distance between elements of the network, the number of ports on each element as well as the total connectivity has received significant attention. These networks can be modeled using both graphs and directed graphs, by representing the communication nodes, microprocessors or the switches by vertices and the links between units by edges. Network communication delay is closely related to the diameter of the representing graph and fault-tolerance of a network is related to the connectivity of the graph. As a consequence of these relationships a great deal of research effort has been spent on the interdependence of minimum and maximum degree, connectivity, and the diameter of graphs and directed graphs [4, 6, 7, 8, 10, 12].

Networks usually require a high degree of regularity, high connectivity and the minimum possible degree for nodes. In modeling fault tolerance characteristics of architectures it is necessary to consider the effect of one or more node failures on the communication behavior of the network. The idea of high connectivity is required to keep the network connected in cases of single or multiple node failures. In such cases, the job being executed by the failed node is usually requeued to another node. This idea is inefficient since the entire job may have to be re-executed. In addition, if the failed node was communicating with several other nodes, as in the execution of a parallel algorithm, then communication of these nodes with the new node may be impossible. In this case all related executing jobs may have to be stopped. Such a catastrophe may be avoided if each node $v$ has 'buddy nodes' that have 'similar adjacency,' so that in case of failure of $v$, its buddy nodes can take up the load and communication behavior of the network will be minimally effected. Therefore a fault tolerant network should not only remain connected, but it should be possible to shift a job from a failed processor to other processors without re-executing the entire job. This dynamic load balancing capability would ensure minimal interruption to the network in case of single or multiple node failures and would increase overall efficiency.

This leads us to define a new property that a network should satisfy in order to sustain single or multiple node failures, that is, each node $v$ in the network should have some nodes that share a large adjacency with $v$. We say that these nodes are *similar* to $v$. A network is called similar if each node in the network satisfies this property. With this in mind we investigate the class of graphs, which model these networks, and which we call similar graphs.

Clearly, there are two somewhat orthogonal ideas involved in the definition of similar graphs. The number of vertices that are similar to a vertex and the degree to which they are similar. A stronger version of similarity arises if two similar vertices have exactly the same adjacency. We call these graphs *strongly similar* graphs.

We define several classes of similar graphs, investigate characterization and properties of these graphs. We also investigate the relationship between regularity, connectivity and similarity. Finally, we consider the problem of augmenting a given graph to make it a similar graph.

In section 2, we investigate the properties of similar graphs and give a characterization for a sub-class. We show that several well known classes of graphs satisfy the similarity property. We consider the relationship between regularity and connectivity. In section 3 we show that augmentation problem associated with similarity is NP-Complete and section 4 deals with characterization of strongly similar graphs.

## 2. Similar graphs.

Let us first give a precise definition of similarity, which leads us to define the class of similar graphs.

**Definition 1.** *Two vertices $x$ and $y$ of a graph $G = (V, E)$ are called $n$-similar if $|Adj(x) \cap Adj(y)| \geq n$.*

**Definition 2.** *A graph $G = (V, E)$ is $(m, n)$-similar if each vertex $v \in V$ is $n$-similar to at least $m - 1$ vertices.*

We will use the notation $S(m, n)$ to denote an $(m, n)$-similar graph.

Figure 1(a) shows an example of an $S(3, 2)$ graph. It may be noted that $S(3, 2)$ is not equivalent to $S(2, 3)$, in fact, the graph in Figure 1(a) is not $S(2, 3)$. The graph shown in Figure 1(b) is not even $S(2, 2)$.

It appears that $S(m, n)$ graphs fail to have enough structure to allow a satisfactory characterization for arbitrary values of $m$ and $n$. Therefore we investigate $(m, n)$-similar graphs for certain values of $m$ and $n$. Every graph is vacuously $S(1, m)$ for all $m$. The characterization of class $S(2, 1)$ is trivial since a graph is $S(2, 1)$ if and only if each vertex is an endpoint of a $P_3$. Therefore we investigate $S(2, 2)$ graphs. This class of graphs is interesting since fault tolerant models frequently assume a single node failure.

*2.1 $S(2, 2)$ graphs.*

In this subsection we present a characterization of $S(2, 2)$ graphs. We show that several well known classes of graphs are included in this class.

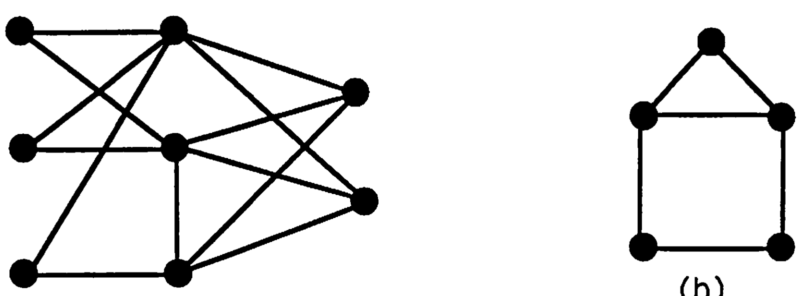


Figure 1
Example of a (3, 2)-similar graph and a non 2-similar graph

Let us define the graph *house* as a $C_5$ with a single chord. An example of a house is shown in Figure 1(b). A 2-connected component of a graph $G$ is called a *block*.

It is easy to see that a house or $k$-cycle $C_k$, $k > 4$ is not an $S(2, 2)$ graph, although it is possible to have a house or a cycle in an $S(2, 2)$ graph, as an induced subgraph. We now present a characterization of $S(2, 2)$ graphs which are both house free and $C_k$-free, $k > 4$. We will assume that all graphs considered in this section have at least four vertices.

**Lemma 1.** *If $G = (V, E)$ is a house free, $C_k$-free, $k \geq 5$ and a 2-connected graph then $G$ is $S(2,2)$ .*

Proof: Assume that $G$ satisfies the hypotheses but fails to satisfy the conclusion. Thus $G$ must contain a vertex $v$ which is not 2-similar to any other vertex. Since $G$ is 2-connected $v$ belongs to at least one cycle. If a vertex belongs to a cycle it belongs to a chordless cycle. Let $k$ be the smallest integer such that $v$ belongs to a chordless $k$-cycle $C$. By hypothesis $k < 5$ and if $k = 4$ then $v$ is 2-similar to a vertex on $C$. Thus $C$ is a triangle. Let $v$, $x_1$ and $x_2$ be the vertices of $C$. Since $G$ has at least four vertices one of the edges, say $e$, of $C$ belongs to a chordless cycle $C'$ such that $C \cap C' = e$. Thus we complete the proof by considering the two cases:

Case I: $(v \in C')$

In this case $C'$ must be a triangle hence $v$ is 2-similar to one of $x_1$ or $x_2$ which ever belongs to $C'$.

Case II: $(v \notin C')$

If $C'$ is a four cycle then $G$ contains a house. If $C'$ is a triangle then $v$ is 2-similar to the vertex of $C'$ other than $x_1$ or $x_2$.

In either case we are led to a contradiction. ∎

**Theorem 1.** *A house free, $C_k$-free $k \geq 5$ graph $G = (V, E)$ is $S(2,2)$ if and only if every vertex $v \in V$ is in a block with at least 4 vertices.*

Proof: For the *if* clause note that if every vertex is in a block of size 4, Lemma 1, implies that each block is an $S(2,2)$ graph and therefore $G$ is an $S(2,2)$ graph. For the *only if* clause assume by contradiction that $G$ is $S(2,2)$ and there exists a vertex $v \in V$ such that there exists no block of size greater than 3 containing $v$. Then $v$ can only be in a block of size 2 or 3. In fact $v$ can be in several such blocks. By assumption $G$ is a $S(2,2)$ graph therefore a vertex $u$ exists that is 2-similar to $v$. This implies that $v$ and $u$ must lie on a four cycle. If all of the blocks in which $v$ lies are of size 2 or 3 then such a 4-cycle cannot exist. This implies that $v$ does not have a 2-similar vertex hence our assumption that $G$ is $S(2,2)$ is contradicted. ∎

This theorem can be used to show that several classes of graphs satisfy the $S(2,2)$ property as proved in following corollaries.

Triangulated, comparability and co-comparability graphs are three important classes of perfect graphs [5]. We now show that significant subclasses of these graphs are contained in the class of $S(2,2)$ graphs.

A graph $G = (V, E)$ is a *triangulated* graph if $G$ contains no chordless cycle $C_k, k \geq 4$. A graph $G = (V, E)$ is a *comparability* graph if each edge $e \in E$ can be assigned a direction in such a way that the resulting oriented graph $(V, F)$ is transitive, that is:

$$uv \in F \text{ and } vw \in F \text{ imply } uw \in F \text{ for all } (u, v, w) \in V.$$

A graph $G = (V, E)$ is called co-comparability graph if $\overline{G}$ is a comparability graph. Where $\overline{G}$ denotes the complement of the graph $G$.

**Corollary 1.** *A 2-connected triangulated graph $G$ is $S(2,2)$.*

Proof: By definition triangulated graphs can not contain a chordless cycle $C_k, k > 4$ thus triangulated graphs are house free since a house contains a $C_4$. In addition if $G$ is 2-connected then $G$ is a house free, $C_k$-free block. ∎

Interval graphs are a subclass of triangulated graphs and it follows that 2-connected interval graphs are also $S(2,2)$. Trees are an interesting sub-class of triangulated graphs which are not $S(2,2)$.

It is known that comparability graphs do not contain an induced $C_{2k+i}, i > 1$. The class of cycle-free comparability graphs are those comparability graphs which do not contain induced chordless cycles of length greater than three. Cycle-free comparability graphs is a well studied class of graphs [5].

**Corollary 2.** *A cycle-free 2-connected comparability graph $G$ is $S(2,2)$.*

A graph $G = (V, E)$ is a permutation graph if and only if $G$ is a comparability and a co-comparability graph. Permutation graphs do not have $C_k, k > 4$ as an induced subgraph [5]. Therefore a permutation graph can have a house graph as an induced subgraph.

**Corollary 3.** *A house free 2-connected permutation graph $G$ is $S(2,2)$.*

*Maximum planar* graphs are planar graphs such that addition of another edge would make the graph non-planar [5].

**Corollary 4.** *Every maximum planar graph is $S(2,2)$.*

A graph $G = (V, E)$ is defined to be split if there exists a partition of vertex set $V$ into an independent set $S$ and a complete set $K$, that is, the induced graph on $K$ is a complete graph. Another characterization of split graphs is in terms of triangulated graphs. A graph $G$ is a split graph if and only if $G$ is a triangulated graph and $\overline{G}$ is a triangulated graph. Let $\delta(G)$ denote the minimum degree of the graph $G$ [5].

**Lemma 2.** *A split graph is $S(2,2)$ if and only if $\delta(G) \geq 2$.*

Proof: First we prove the 'if' clause. Suppose $G$ is a split graph such that $\delta(G) \geq 2$. Note that if $S = \emptyset$ then $G$ is a complete graph and hence $S(2,2)$. If $S \neq \emptyset$ then $K$ contains at least two vertices, but then it follows from the fact that $\delta(G) \geq 2$ that $|S| \geq 2$.

Case I: $(|K| = 2)$

In this case any pair of vertices in $S$ are 2-similar, and the two vertices in $K$ are 2-similar.

Case II: $(|K| = 3)$

Every vertex in $S$ is adjacent to at least two vertices of a triangle in $K$ thus is contained in a 4-cycle along with the three vertices of $K$.

Case III: $(|K| > 3)$

The same argument as that given in case 2 applies to the vertices in $S$ and since the induced graph on $K$ is a complete graph it is $S(2,2)$.

In any $S(2,2)$ graph the degree of a vertex must be at least two. Thus the 'only if' clause is clear. ∎

*2.2 Exact $S(2,2)$-graphs.*

It is possible that a node $v$ in an $(m,n)$-similar graph may have adjacent vertices that are not adjacent to any vertex which is $n$-similar to $v$. In this case a failure of $v$ may still lead to a communication breakdown. To circumvent this situation we may require a stronger condition as defined below.

**Definition 3.** *A graph $G = (V, E)$ is exact $(m,n)$-similar if each vertex $v \in V$ is $n$-similar to exactly $m-1$ vertices.*

We will say that an $(m,n)$-similar graph is $ES(m,n)$. We investigate the interaction of regularity and connectivity of $ES(2,2)$ graphs. We show that $ES(2,2)$ are highly structured graphs. We also show that if an $ES(2,2)$ graph is 3-regular then it is 2-connected. These properties are interesting from a network design point of view.

**Lemma 3.** *If $G = (V, E)$ is a $ES(2,2)$ graph then every vertex $v \in V$ lies in a unique four cycle.*

Proof: It is easy to see that every vertex and its 2-similar vertex must lie in a four cycle, so all we have to show is the uniqueness of such a cycle. Suppose by contradiction that $G = (V, E)$ is a $ES(2,2)$ graph and there exists a vertex $v \in V$ such that $v$ is in two four cycles then we have three cases to consider, depending upon the number of edges the two four cycles have in common. Let $C_1$ and $C_2$ be the two distinct four cycles in which $v$ lies.

Case I: ($C_1$ and $C_2$ have no edges in common)

If $C_1 = v, u_1, u_2, u_3$ and $C_2 = v, w_1, w_2, w_3$ has no edge in common then both $u_2$ and $w_2$ are 2-similar to $v$ as shown in Figure 2(a).

Case II: ($C_1$ and $C_2$ have one edge in common)

If $C_1 = v, u_1, u_2, u_3$ and $C_2 = v, w_1, w_2, w_3$ have one edge in common then either $u_1 = w_1$ or $u_3 = w_3$ in both cases $u_2$ and $w_2$ are

2-similar to $v$ as shown in Figure 2(b).

Case III: ($C_1$ and $C_2$ have two edges in common)

If $C_1 = v, u_1, u_2, u_3$ and $C_2 = v, w_1, w_2, w_3$ have two edges in common, then these cycles have three vertices in common. Let $v, u_1, u_3$ be the common vertices and let $u_1 = w_1$ and $u_3 = w_3$ in this case $u_2$ and $w_2$ are 2-similar to $v$ as shown in Figure 2(c). In other sub cases when $v$ in not the *middle* vertex as shown in Figure 2(c), the middle vertex will always have at least two vertices to which it is 2-similar.

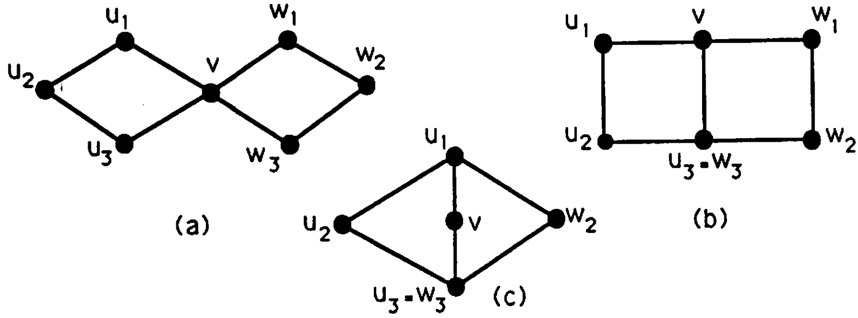


Figure 2

Every vertex of a $ES(2,2)$ graph is in a unique four cycle.

It is important to note that two four cycles cannot have three edges in common without having the fourth edge in common. In each case we have shown that there exist more than one vertex 2-similar to $v$ thus contradicting our assumption that $G$ is a $ES(2,2)$ graph. ∎

The lemma above leads immediately to the following corollary.

**Corollary 5.** *An $ES(2,2)$ graph has $4n$ vertices.*

**Theorem 2.** *If a graph $G$ is $ES(2,2)$ and 3-regular then it is 2-edge connected.*

Proof: Suppose by contradiction that $G$ is $ES(2,2)$ and 3-regular but not 2-edge connected. This implies that a cut edge $e = (u, v)$ exists, removal of which disconnects $G$ into connected components $G_u$ and $G_v$ containing $u$ and $v$ respectively. We claim that both $G_u$ and $G_v$ must be $ES(2,2)$. Let $w$ be the vertex 2-similar to $u$ in $G$. Since $u$ and $w$ must lie in a unique four cycle, $w$ must also be in $G_u$. The same argument applies to every vertex of $G_u$ and similarly for $G_v$. This proves our claim that $G_u$ is $ES(2,2)$. As $G_u$ is $ES(2,2)$ then by Corollary 5 it must have $4t$ vertices, where $t$ is a positive integer. But every vertex in $G_u$ has degree three except $u$ which has degree 2. This implies that $G_u$ has an odd number of vertices with odd degree. This contradicts the fact that every simple graph has an even number of vertices with odd degree. ∎

## 3. NP-Completeness of similar augmentation.

An existing network may fail to satisfy a given similarity property. A natural problem to consider is to augment this network by adding edges to obtain a network that does satisfy the given similarity property. In this section we show that it is unlikely that an efficient algorithm exists to answer this problem by showing that this problem is NP-Complete. The similarity augmentation (AS) problem is defined as follows:

$AS(m,n)$ problem:

Instance: A graph $G = (V, E)$ and positive integers $K$, $m$ and $n$.

Question: Is there a set of edges $E'$ such that

$$|E'| \leq K \text{ and } G' = (V, E \cup E') \text{ is } S(m,n)?$$

The following theorem shows that a restriction of the $AS(m,n)$ problem is NP-Complete which implies that the $AS(m,n)$ problem is NP-Complete.

**Theorem 3.** *The $AS(2,2)$ problem is NP-Complete.*

Proof: We show that the $AS(2,2)$ problem is NP-Complete by giving a transformation from $3SAT$ to $AS(2,2)$ [2]. Let $\{u_i \mid i = 1, \ldots, r\}$ and $\{c_j \mid j = 1, \ldots, r\}$ be the variables and clauses, respectively, of an instance of $3SAT$. Define an instance $G$ and $K$ of $AS(2,2)$ as follows: For each variable $u_i$ $i = 1, \ldots, r$ construct the graph shown in Figure 3(a) and for each clause $c_j$, $j = 1, \ldots, s$ use a $K_2$ labled as shown in Figure 3(b).

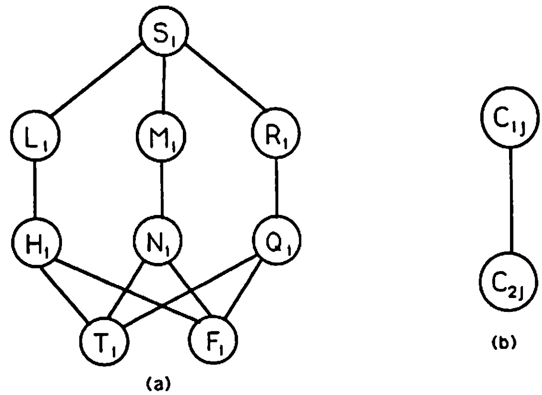


Figure 3
Components used in transformation.

The different graphs are connected by adding the edges:

$$\{S_iC_{1j} \mid 1 \leq i \leq r,\ 1 \leq j \leq s \text{ and either } u_i \text{ or } \overline{u}_i \text{ is in the clause } c_j\}$$

$$\{T_iC_{2j} \mid 1 \leq i \leq r,\ 1 \leq j \leq s \text{ and } u_i \text{ is in the clause } c_j\}$$

and

$$\{F_iC_{2j} \mid 1 \leq i \leq r,\ 1 \leq j \leq s \text{ and } \overline{u}_i \text{ is in the clause } c_j\}$$

and we define $K = r$, the number of variables. An example is shown in Figure 4 for the clause $c_i = u_1 u_2 \overline{u}_3$.

Suppose there exists an assignment

$$\theta: \{u_i \mid 1 \leq i \leq r\} \rightarrow \{\text{True, False}\}$$

such that each clause has the value True for this assignment. We use $\theta$ to add $K = r$ edges to $G$, that is, the edges

$$E' = \{S_iX \mid 1 \leq i \leq r \text{ and } X = \begin{cases} T_i & \text{if } \theta(u_i) = \text{True} \\ F_i & \text{if } \theta(u_i) = \text{False} \end{cases}.$$

The addition of the single edge $S_iT_i$ or $S_iF_i$ causes the subgraph $G_i$, associated with the variable $u_i$, to be $S(2,2)$. This is so because every vertex in $G_i$ is in a four cycle. Since $\theta$ insures that each clause has at least one true literal we see that for each $1 \leq j \leq s$, $c_{1j}$ and $c_{2j}$ have 2-similar vertices. Let $i$ be the index of the literal contained in $c_j$ which has value true. If the edge $S_iT_i$ is added then $C_{1j}$ is 2-similar to $T_i$, on the other hand, if the edge $S_iF_i$ is added then $C_{1j}$ is 2-similar to $F_i$. $C_{2j}$ is 2-similar to $S_i$ irrespective of the addition of $S_iT_i$ or $S_iF_i$. Therefore the graph $G = (V, E \cup E')$, and $|E| \leq K$ is $S(2,2)$.

Now we show that given at most $K$ edges whose addition makes $G$ an $S(2,2)$ graph guaranties a satisfying truth assignment. Suppose that there exists a set of edges $E'$ such that $|E'| \leq K$ and the addition of these edges to $G$ results in a graph $G'$ which is $S(2,2)$. First we will show that $|E'| \leq K$. Let $G_i$ be the subgraph associated with the variable $u_i$. Since $S_i$, $L_i$, $M_i$, and $R_i$ fail to have 2-similar vertices in $G$, at least two of the vertices of $G_i$ must be endpoints of edges in $E'$. Since this is true for each $1 \leq i \leq r$ it follows that $|E'| \geq K$. Thus $|E'| = K$.

Let us now show that for each $i$ exactly one of the edges $S_iT_i$ or $S_iF_i$ is in $E'$. By the above argument we know that for each $i$ the subgraph $G_i$ has exactly two vertices which are endpoints of edges in $E'$. If these vertices are not the endpoints of a single edge in $E'$ then at least one of $S_i$,$L_i$, $M_i$, or $N_i$ will fail to have a 2-similar vertex. Thus for each $1 \leq i \leq r$, the subgraph $G_i$ must receive exactly one of the added edges $x_i$ of $E'$. The addition of $x_i$ to $G_i$ must result in an $S(2,2)$ graph $G_i'$. It is clear by inspection that there are only two edges $S_iT_i$ or $S_iF_i$ whose addition to $G_i$ results in an $S(2,2)$ graph.

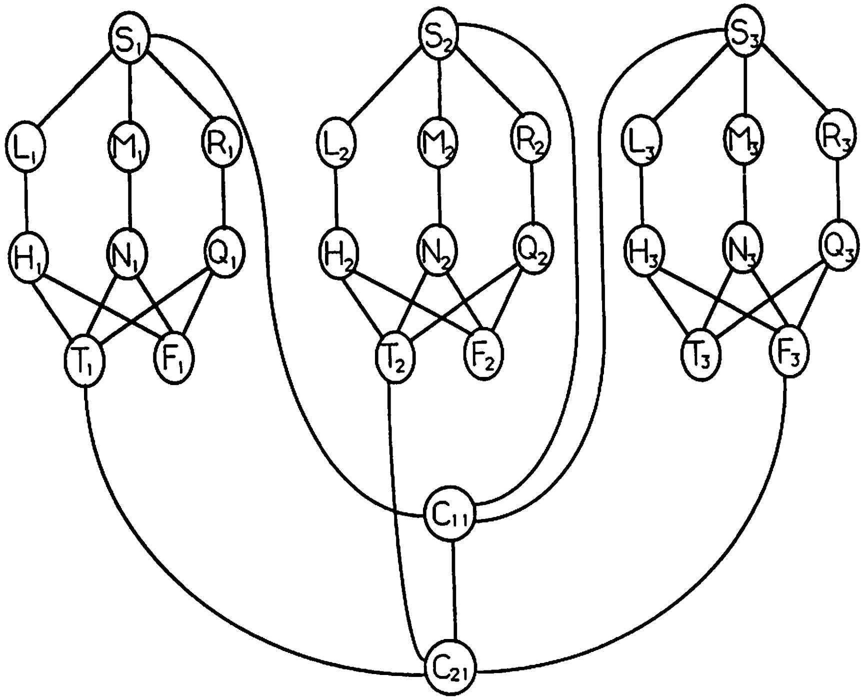


Figure 4
Graph for clause $c_i = u_1 u_2 \overline{u}_3$.

We construct the assignment

$$\theta\colon \{u_i \mid i = 1, \ldots, r\} \to \{\text{True, False}\}$$

defined by

$$\theta(u_i) = \begin{cases} \text{True} & \text{if } S_iT_i \in E' \\ \text{False} & \text{if } S_iF_i \in E' \end{cases}.$$

Since $G'$ is $S(2,2)$ each of $C_{2j}$ and $C_{1j}$, $j = 1, \ldots, s$ must be contained in a four cycle, which must contain an edge of $E'$, that is, $S_iT_i$ or $S_iF_i$ for some $i$ such that $u_i$ or $\overline{u}_i$ is a literal in $c_j$. Thus $\theta$ satisfies the clauses in the instance of $3\,SAT$. Since $3\,SAT$ is NP-Complete [2], it follows that $AS(2,2)$ is NP-Complete. ∎

## 4. Strongly similar graphs.

In some networks a dummy node is added for each node that is critical for network operation. In case of failure of critical nodes, the dummy nodes take over. In order to model these networks we define a stronger version of similarity. In this section we present a complete characterization of these graphs.

**Definition 4.** *Two vertices $x$ and $y$ of a graph $G = (V, E)$ are strongly similar if $x$ and $y$ have the same adjacency set, that is, $Adj(x) = Adj(y)$.*

**Definition 5.** *A graph $G = (V, E)$ is strongly $m$-similar if each vertex $v \in V$ is strongly similar to at least $m - 1$ vertices.*

We use the notation $SS(m)$ to denote a strongly $m$-similar graph. Given a graph $H = (V_H, E_H)$ and a function $w: V_H \rightarrow Z^+$ define the graph $H^w = (V_H^w, E_H^w)$ to have vertices:

$$V_H^w = \{\overline{K}_{w(v_i)} \mid v_i \in V_H\}$$

and edges:

$$E_H^w = \{xy \mid x \in \overline{K}_{w(v_i)}, y \in \overline{K}_{w(v_j)} \text{ and } v_iv_j \in E_H\}$$

where $\overline{K}_r$ denotes the complement of the complete graph on $r$ vertices, that is, $\overline{K}_r$ is a set of $r$ isolated vertices.

**Theorem 4.** *A graph $G = (V, E)$ is $SS(m)$ if and only if $G$ is isomorphic to $H^w$ where $H = (V_H, E_H)$ is an arbitrary graph and $w(v) \geq m$ for every $v \in V_H$.*

Proof: Let $G = (V, E)$ be a graph which is $SS(m)$ for a positive integer $m$. Define the relation "$\sim$" on $V$ by $x \sim y$, if $x$ and $y$ have the same adjacency set. Clearly, "$\sim$" is an equivalence relation on $V$ and thus partitions $V$ into equivalence classes, $V_i$ $i = 1, \ldots, n$ for some $n$. If $x \sim y$ then $xy \notin E$.

Claim: If there exists an edge between $u \in V_i$ and $v \in V_j$, $i \neq j$ then the induced graph on $V_i \cup V_j$ is $K_{|V_i|,|V_j|}$.

Suppose $u \in V_i$, $v \in V_j$ and $uv \in E$, if $x \in V_i$ and $y \in V_j$ then $xv \in E$ since $x \sim u$. Since $xv \in E$ and $v \sim y$ it follows that $xy \in E$. Thus we have proven the claim.

Define $w$: $V \rightarrow Z^+$ by $w(v) = |V_i|$ where $v \in V_i$. Let $H$ be the graph with vertices $\{V_i \mid i = 1, \ldots, n\}$ and edges $V_iV_j$ if there is an edge in $G$ which is incident to a vertex in $V_i$ and a vertex in $V_j$. By the claim we see that $G$ is isomorphic to $H^w$. The converse is clear from the definition of $H^w$. ∎

The example in Figure 5 shows that the representation given in Theorem 1 is not unique. $H_1$ and $H_2$ are non-isomorphic graphs but the transformation results in isomorphic graphs due to the different weighting functions used.

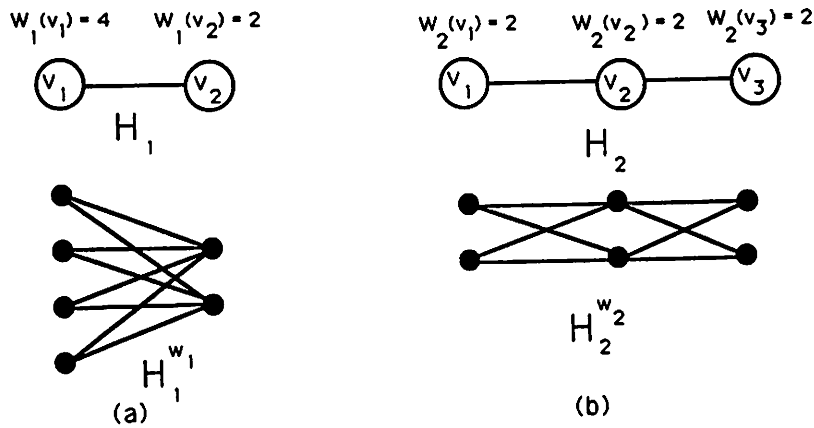


Figure 5
Graphs $H$ and $H^w$.

*4.1 Exact SS-graphs.*

In this section, we introduce an additional restriction on the class of strongly $n$-similar graphs and present a characterization for this class of graphs. As in the case of similar graphs, failure of a node may lead to failure of the communication pattern because strong similarity also does not guarantee that all adjacent nodes are covered. We start by giving a strengthened definition of strong similarity.

**Definition 6.** *A graph $G = (V, E)$ is exactly strongly $m$-similar if each vertex $v \in V$ is strongly similar to exactly $m - 1$ vertices.*

We use the notation $ESS(m)$ to denote an exact strongly $m$-similar graph. We now present a complete characterization of these graphs. This characterization is analogous to the characterization of strongly $m$-similar graphs.

**Theorem 5.** *A graph $G = (V, E)$ is $ESS(m)$ if and only if $G$ is isomorphic to $H^w$ where $H = (V_H, E_H)$ is a graph with no two strongly similar vertices and $w(v) = m$ for every $v \in V_H$.*

Proof: Let $G = (V, E)$ be a graph which is $ESS(m)$. Let the equivalence relation "$\sim$" on $V$ be as defined in the proof of Theorem 4, that is, $x \sim y$ if they have the same adjacency set. In this case the equivalence classes for "$\sim$", must each contain exactly $m$ elements and the graph induced on the union of a pair of equivalence classes must be $K_{m,m}$ or be empty. Thus as in the proof of Theorem 4, $G$ is isomorphic to $H^w$, but here $w(v) = m$ for each $v \in V_H$. The fact that the equivalence classes of $V$ must have the same cardinality implies that $H$ does not contain strongly similar vertices.

Conversely, suppose $H = (V_H, E_H)$ is a graph which fails to contain a pair of strongly similar vertices and $w\colon V_H \rightarrow Z^+$ such that $w(v) = m$ for each $v \in V_H$. Clearly every vertex of $H^w$ is strongly similar to at least $m-1$ vertices. If a vertex $v$ is strongly similar to $n \geq m$ vertices then by the definition of $H^w$, $H$ must have two vertices say $x$ and $y$ such that $\overline{K}_{w(x)}$ and $\overline{K}_{w(y)}$ both contain vertices which are strongly similar to $v$. But this implies that $x$ and $y$ are strongly similar in $H$, which is contrary to the hypothesis. ∎

## 5. Conclusion.

In this paper, we have introduced the concept of similar graphs. Networks with good similarity characteristics have the capability to respond efficiently to single or multiple node failures.

We have characterized several classes of these graphs. Several well known classes of graphs were shown to satisfy the similarity property. We also proved that the problem of augmenting a graph by adding edges to become $(m, n)$-similar is NP-Complete and remains so even if $m = n = 2$.

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