On Matching and Chromatic Properties of Circulants

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Abstract. It is shown that the basis graphs of every family of circulants are characterized by their matching polynomials. Explicit formulas are also given for their matching polynomials. From these results, the analogous formulas for the chromatic polynomials of the complements of the basis graphs, are obtained. It is shown that a basis graph of a family of circulants is chromatically unique if and only if it is connected. Also, some interesting results of a computer investigation are discussed and conjectures are made.

1. Introduction.

The graphs considered here are finite and contain no loops and no multiple edges. Let G be such a graph and p denote the number of nodes in G. Let $\overline{G}$ denote the complement of G. A *matching* in G is a spanning subgraph of G whose components are nodes and edges only. A *k-matching* is a matching with exactly k edges. A *defect-d matching* is a matching with exactly d nodes. A *perfect matching* is a defect-0 matching. The *matching polynomial* of G is the polynomial

$$M(G;\mathbf{w}) = \sum_k a_k w_1^{p-2k} w_2^k,$$

where a_k is the number of k-matchings in $G, \mathbf{w} = (w_1, w_2)$, where w_1 and w_2 are indeterminates associated with a node and an edge respectively, and the summation is taken over all integers k such that $0 \leq k \leq \lfloor p/2 \rfloor$. The notation $\lfloor x \rfloor$ means the greatest integer less than or equal to x. We refer the reader to Farrell [4] for the basic properties of matching polynomials.

The *chromatic polynomial* of a graph G counts the number of proper colorings of the nodes of G with λ colors, where λ is an indeterminate. A proper coloring is a coloring in which adjacent nodes receive different colors. We will denote the chromatic polynomial of G by $P(G; \lambda)$. Also throughout this paper we will assume that $P(G; \lambda)$ is expanded in the *falling factorial basis*, that is,

$$P(G; \lambda) = \sum_k b_k(\lambda)_k,$$

where $(\lambda)_k = \lambda(\lambda-1)(\lambda-2)\ldots(\lambda-k+1)$ and the summation is taken over all integers k such that $1 \leq k \leq p$. We refer the reader to Read [11] for the basic properties of chromatic polynomials.

The following definition is taken from Boesch and Tindell [1]. The *circulant graph* $C_p\langle a_1, a_2, \ldots, a_k\rangle$ is a graph with p nodes where $0 < a_1 < a_2 < \ldots < a_k \leq \lfloor p/2 \rfloor$, and nodes $i \pm a_1,\ i \pm a_2, \ldots, i \pm a_k \pmod{p}$ are adjacent to node i for each i, where $1 \leq i \leq p$. For the purposes of this paper, we will denote by B_r the circulant $C_p\langle r\rangle$.

It is clear from the definition, that the graphs B_r $(1 \leq r \leq \lfloor p/2 \rfloor)$ are edge disjoint. Also every circulant on p nodes can be formed by including the edges of a unique subset of the set $(B_1, B_2, \ldots, B_n)$ where $n = \lfloor p/2 \rfloor$. We therefore refer to this set of circulants as the basis for all circulants on p nodes.

We say that a graph G is *characterized* by its matching polynomial if and only if whenever $M(G;\boldsymbol{w}) = M(H;\boldsymbol{w})$, for a graph H, then G is isomorphic to H. For brevity we will say that G is *matching unique*. The term *chromatically unique* is analogously defined by using the chromatic polynomial instead of the matching polynomial. Graphs G and H are called *co-matching* when $M(G;\boldsymbol{w}) = M(H;\boldsymbol{w})$.

In this paper, we show that the basis graphs $B_1, B_2, \ldots, B_n$ for each positive integer p, are matching unique. Also, we give explicit formulas for the matching polynomials of the basis graphs. From these we deduce explicit formulas for the chromatic polynomials of the complements of the basis graphs. Finally we give the result of a computer investigation which seems to indicate that the family of circulant graphs are rich in chromatically unique graphs. It appears that every circulant graph is either matching unique or chromatically unique or both.

Let C_p denote the cycle with p nodes. (When $p = 1$ or 2, we take C_p to be a node or an edge respectively. Cycles with more than two nodes are called *proper cycles*). A graph consisting of components $H_1, H_2, \ldots, H_k$ will be denoted $H_1 \cup H_2 \cup \ldots \cup H_k$.

2. Some preliminary results on circulants.

In [1] Boesch and Tindell remark that not all point-symmetric graphs are circulants. However every point-symmetric graph with a prime number of nodes is a circulant. They point out that the cube $Q_3 (= K_2 \times C_4$, where x denotes the *cartesian product* (see Harary [10], p. 22)) is the smallest point-symmetric graph that is not a circulant. Boesch and Tindell prove that the circulant $C_p\langle a_1, a_2, \ldots, a_k\rangle$ is connected if and only if $gcd(a_1, a_2, \ldots, a_k, p) = 1$. The notation $gcd(d_1, d_2, \ldots, d_s)$ means the greatest common divisor of the integers $d_1, d_2, \ldots,$ and d_s. From this connectivity result, it follows that if $gcd(a_1, a_2, \ldots, a_k, p) \neq 1$ then $C_p\langle a_1, a_2, \ldots, a_k\rangle$ is not chromatically unique because of the following result in Chia [3]. Chia proved that if G has two blocks each containing at least 3 nodes then G is not chromatically unique.

A *prism* is the cartesian product of an edge (K_2) with a proper cycle (C_p where $p \geq 3$). Thus, $K_2 \times C_p$ denotes a prism with $2p$ nodes. When p is an odd integer, $K_2 \times C_p$ is the circulant $C_{2p}\langle 2, p\rangle$. When p is an even integer, $K_2 \times C_p$ is not a circulant because $C_{2p}\langle 2, p\rangle$ is not connected.

3. Some preliminary results on matching polynomials.

The following lemma is Theorem 5 in [4]. It is easy to prove.

Lemma 1. *(The Component Theorem) Let G be a graph with components $G_1, G_2, \ldots, G_k$. Then*

$$M(G; \boldsymbol{w}) = \Pi_i M(G_i; \boldsymbol{w}).$$

The following lemma can be easily deduced from the definition of the matching polynomial.

Lemma 2. *Let G be a graph with p nodes. Then*

(i) *The highest power of w_1 in $M(G; \boldsymbol{w})$, is the number of nodes in G.*
(ii) *The coefficient of $w_1^{p-2} w_2$ in $M(G; \boldsymbol{w})$, is the number of edges in G.*
(iii) *For p even, the coefficient of $w_2^{p/2}$ in $M(G; \boldsymbol{w})$, is the number of perfect matchings in G.*
(iv) *The coefficient of $w_1^d w_2^{(p-d)/2}$ in $M(G; \boldsymbol{w})$, is the number of defect-d matchings in G.*

The following result was proven in Farrell and Guo [5].

Lemma 3. *If a graph is regular of valency d, then any co-matching graph is also regular of valency d.*

4. Matching properties of the union of isomorphic cycles.

The following lemma can be easily proved. Its proof is left to the reader.

Lemma 4. *Let F be the family of circulants with p nodes. Then the basis for F is $(B_1, B_2, \ldots, B_n)$ where $n = \lfloor p/2 \rfloor$ and B_r is the union of $d = gcd(r, p)$ copies of the cycle $C_{p/d}$.*

The following lemma is taken from [4].

Lemma 5.

$$M(C_p; \boldsymbol{w}) = \sum_{k=0}^{\lfloor p/2 \rfloor} \frac{p}{p-k} \binom{p-k}{k} w_1^{p-2k} w_2^k.$$

By the Component Theorem (Lemma 1), we immediately obtain the following result.

Theorem 1. *Let* $G = C_{r_1} \cup C_{r_2} \cup \ldots \cup C_{r_n}$. *Then* $M(G;\boldsymbol{w}) = \Pi_{i=1}^{n} M(C_{r_i};\boldsymbol{w})$. *In the special case when* $r_i = q$ *for all* i, *we obtain the following result.*

Corollary 1.1. *Let* $G = C_p \cup C_p \cup \ldots \cup C_p$ *(n times). Then* $M(G;\boldsymbol{w}) = [M(C_p;\boldsymbol{w}]^n$.

The following is a general result about matching polynomials.

Theorem 2. *The matching polynomial characterizes the unions of copies of a cycle.*

Proof: We consider improper cycles first, then proper cycles.

Improper cylces.

Let $G = K_1 \cup K_1 \cup \ldots \cup K_1$ (n times). Then G is trivially characterized by its matching polynomial.

Let $G = K_2 \cup K_2 \cup \ldots \cup K_2$ (n times). Then we have

$$M(G;\boldsymbol{w}) = (w_1^2 + w_2)^n = \sum_{r=1}^{n} \binom{n}{r} w_1^{2r} w_2^{n-r}.$$

Let H be a graph such that $M(H;\boldsymbol{w}) = M(G;\boldsymbol{w})$. By Lemma 2, we obtain

(i) H has $2n$ nodes,
(ii) H has n edges, and
(iii) H has one perfect matching.

Now, the perfect matching in H is a spanning subgraph of H. Since it contains n edges it contains all the edges of H. Hence the perfect matching constitutes all of H. It follows that H is isomorphic to G. Hence G is characterized by its matching polynomial.

Proper cylces.

Let $G = C_p \cup C_p \cup \ldots \cup C_p$ (n times). where $p > 2$. It has been shown by Farrell and Guo [5] that C_p is characterized by its matching polynomial. Therefore we assume that $n > 1$. We consider two cases (i) p even and (ii) p odd.

p even.

Let H be a graph such that $M(H;\boldsymbol{w}) = M(G;\boldsymbol{w})$. Since G is regular of valency 2, it follows (by Lemma 3) that H is also regular of valency 2. This implies that H is either a cycle or a union of cycles. Since p is even, C_p has two perfect matchings. Therefore, we can write

$$M(Cp;\boldsymbol{w}) = w_1^p + pw_1^{p-2}w_2 + \ldots + 2w_2^{p/2}.$$

Therefore,

$$M(G;\boldsymbol{w}) = \left[w_1^p + pw_1^{p-2}w_2 + \ldots + 2w_2^{p/2}\right]^n = M(H;\boldsymbol{w}).$$

By extracting the relevant terms we obtain (from Lemma 2) the following information about H.

(i) H has pn nodes.
(ii) H has pn edges.
(iii) H has 2^n perfect matchings (from the coefficient of $w_2^{pn/2}$).
(iv) H has $n2^{n-1}p^2/4$ defect-2 matchings (from the coefficient of $w_1^2 w_2^{(pn/2)-1}$).

Since $n > 1$, we have $2^n > 2$. Therefore H has more than 2 perfect matchings. Thus, H cannot be a cycle. Hence, H must be a union of cycles. Since H has perfect matchings, all of the cycles must be even. Let $H = C_{r_1} \cup C_{r_2} \cup \ldots \cup C_{r_n}$, where r_i (> 2) are even integers for $i = 1, 2, \ldots, n$.

Then H has $\sum_{i=1}^{n} r_i$ nodes. Since H has pn nodes, we have

$$\sum_{i=1}^{n} r_i = pn. \tag{1}$$

Now, we can count the number of defect-2 matchings in H. First of all, we observe that a defect-2 matching in H can only be formed by a defect-2 matching in some C_{r_i} and perfect matchings in all of the other cycles. The number of defect-2 matchings in C_{r_i} is $r_i^2/4$, from Lemma 5, with $k = (p-2)/2$. The number of perfect matchings in the $(n-1)$ other cycles is 2^{n-1}. Hence the number of defect-2 matchings in H is equal to

$$\sum_{i=1}^{n} \frac{r_i^2}{4} 2^{n-1} = \frac{2^{n-1}}{4} \sum_{i=1}^{n} r_i^2.$$

Hence from property (iv) of H, we have

$$\begin{aligned} \frac{2^{n-1}}{4} \sum_{i=1}^{n} r_i^2 &= 2^{n-1} n \frac{p^2}{4}. \\ \Rightarrow \sum_{i=1}^{n} r_i^2 &= np^2. \end{aligned} \tag{2}$$

It can be easily shown (by elementary algebra) that Equations (1) and (2) have the unique solution $r_1 = r_2 = \ldots = r_n = p$. Hence $H = C_p \cup C_p \cup \ldots \cup C_p = G$. It follows that G is characterized by its matching polynomial.

p odd.

In this case, we can write

$$M(G; w) = \left[w_1^p + p w_1^{p-2} w_2 + \ldots + p w_1 w_2^{(p-1)/2} \right]^n. \tag{3}$$

Let H be a graph such that $M(H;\mathbf{w}) = M(G;\mathbf{w})$. Then Equation (3) yields the following:

(i) H has pn nodes.
(ii) H has pn edges.
(iii) H has no perfect matchings.
(iv) H has p^n defect-n matchings.

Again, from Lemma 3, we obtain that H must be regular of valency 2. Therefore, H is either a cycle or a union of cycles. If H is a cycle then H is isomorphic to C_{pn} since H has pn nodes. Hence the number of defect-1 matchings in H is pn. However, from Equation (3) the coefficient of $w_1 w_2^{(pn-1)/2}$ is zero. Therefore, H is not isomorphic to C_{pn}. It follows that H is a union of cycles. Let $H = C_{r_1} \cup C_{r_2} \cup \ldots \cup C_{r_n}$. Then

$$\sum_{i=1}^{n} r_i = pn. \tag{4}$$

Since the lowest power of w_1 within the square brackets of Equation (3) is 1, it follows that the lowest power of w_1 in the expansion of the right hand side of Equation (3) is w_1^n; from property (iii) of H, this term occurs with nonzero coefficient p^n. Suppose that some r_i is even. Then H will contain a matching with $n-1$ nodes or less (if other r_i's are also even). Thus $M(H;\mathbf{w})$ will contain a nonzero term involving w_1^k, for some $k < n$. This is impossible. Therefore, all of the r_i's must be odd.

Now, a defect-n matching in H can occur in only one way. Each cycle C_{r_i} must be covered with a defect-1 matching. The number of defect-1 matchings in C_{r_i} is r_i. Therefore the number of defect-n matchings in H is

$$\Pi_{i=1}^{n} r_i = p^n \quad \text{(from property (iv))}. \tag{5}$$

It can be shown that Equations (4) and (5) have the unique solution $r_i = p$, for $i = 1, 2, \ldots, n$. Therefore, $H = C_p \cup C_p \cup \ldots \cup C_p = G$. Hence G is characterized by its matching polynomial. This completes the proof of the theorem. ∎

Lemma 4 and Theorem 2 yield the following result.

Theorem 3. *Let F be a family of circulants. Then all of the basis graphs of F are matching unique.*

5. Matching and chromatic polynomials of basis complements.

A connection between the matching polynomial and the chromatic polynomial is given in the following lemma which is taken from Farrell and Whitehead [7]. The if part is given in Frucht and Giudici [8].

Lemma 6. *Let G be a graph with p nodes. Then $P(\overline{G};\lambda) = M(G;w')$, where w' means that w^k in $M(G;(w,w))$ is replaced by $(\lambda)_k$ and dually, $M(G;\boldsymbol{w}) = P(\overline{G};\lambda')$ where λ' means that $(\lambda)_k$ is replaced by the monomial $w_1^{2k-p}w_2^{p-k}$ if and only if G has no triangles.*

It is clear from the definition of the basis that B_1 is the cycle C_p. Therefore, Lemmas 5 and 6 yield the following result.

Theorem 4.

$$M(B_1;\boldsymbol{w}) = \sum_k \left(\frac{p}{p-k}\right)\binom{p-k}{k} w_1^{p-2k}w_2^k \text{ and for } p > 3 \text{ and } n = \lfloor p/2 \rfloor,$$

$$P(\overline{B}_1;\lambda) = P(C_p\langle 2,3,\ldots,n\rangle;\lambda) = \sum_k \left(\frac{p}{p-k}\right)\binom{p-k}{k}(\lambda)_{p-k}.$$

From Lemma 4, we know that $B_r = C_{p/d} \cup C_{p/d} \cup \ldots \cup C_{p/d}$ (d times) where $d = gcd(p,r)$. Hence we have the following result.

Theorem 5. *For $1 < m \leq n$,*

$$M(B_m;\boldsymbol{w}) = \left[\sum_k \left(\frac{p/d}{(p/d)-k}\right)\binom{(p/d)-k}{k} w_1^{p-2k}w_2^k\right]^d,$$

where $d = gcd(p,m)$. Also, if B_m is triangle-free, then

$$P(\overline{B}_m;\lambda) = P(C_p\langle 1,2,\ldots,m-1,m+1,\ldots,n\rangle;\lambda)$$
$$= \left[\sum_k \left(\frac{p/d}{(p/d)-k}\right)\binom{(p/d)-k}{k}(\lambda)_{p-k}\right]^d.$$

In the particular case when p is even and $n = p/2$, we have the following corollary.

Corollary 5.1.

$$M(B_n;\boldsymbol{w}) = M(K_2 \cup K_2 \cup \ldots \cup K_2;\boldsymbol{w}) = \sum_j \binom{n}{j} w_1^{2j}w_2^{n-j} \text{ and}$$

$$P(\overline{B}_n;\lambda) = P(C_p\langle 1,2,\ldots,n-1\rangle;\lambda) = \sum_j \binom{n}{j}(\lambda)_{n+j}.$$

Theorem 6. *For $p \geq 3$, a circulant basis graph B_r is chromatically unique if and only if $gcd(r,p) = 1$.*

Proof: Let $d = gcd(r,p)$. If $d = 1$ then B_r is C_p by Lemma 4. In [2], Chao and Whitehead proved that the cycle C_p is chromatically unique for all $p \geq 3$. If

$d \neq 1$ then B_r is the union of d copies of $C_{p/d}$ by Lemma 4. If $p/d = 2$ then B_r is the union of d copies of K_2 which was shown to be not chromatically unique by Loerinc and Whitehead in [10]. If $p/d \geq 3$ then B_r was shown to be not chromatically unique by Chia in [3]. ∎

6. Computer results on chromatic uniqueness.

A computer investigation was made into the chromatic uniqueness of circulants having p nodes where $3 \leq p \leq 8$. Of the 30 nonisomorphic circulants investigated, 23 circulants were found to be chromatically unique. In Table 1, we list the 7 circulants found to be not chromatically unique. Three of these 7 circulants are forests all of whose edges form a perfect matching. In [10], Loerinc and Whitehead showed that these forests are not chromatically unique. Another three of these 7 circulants are graphs which contain two blocks each containing at least 3 nodes. By Chia's theorem stated in Section 2, these graphs are not chromatically unique. The last of these 7 circulants is $C_8\langle 1,3,4\rangle$ which has the same chromatic polynomial as the complement of the disconnected graph shown in Figure 1.

Table 1

p	Number of circulants	Not chromatically unique
3	1	none
4	3	$B_2 = K_2 \cup K_2$
5	2	none
6	7	$B_2 = C_3 \cup C_3$ $B_3 = K_2 \cup K_2 \cup K_2$
7	3	none
8	14	$B_2 = C_4 \cup C_4$ $B_4 = K_2 \cup K_2 \cup K_2 \cup K_2$ $C_8\langle 2,4\rangle = K_4 \cup K_4$ $C_8\langle 1,3,4\rangle = \overline{C_4 \cup C_4}$

Figure 1

In [13], Whitehead studied the chromatic polynomials of chorded cycles. In our notation a *chorded cycle* is $C_p\langle 1,2\rangle$. Read [12] used a computer to investigate the chromatic uniqueness of chorded cycles. He found $C_p\langle 1,2\rangle$ to be chromatically unique for $5 \leq p \leq 9$. He conjectured that $C_p\langle 1,2\rangle$ is chromatically unique for all $p \geq 5$.

We conjecture that the complement of the cycle C_p, denoted $C_p\langle 2,3,\ldots,n\rangle$ ($n = \lfloor p/2 \rfloor$) is chromatically unique for all $p \geq 5$. This conjecture is true for $5 \leq p \leq 8$ because $C_5\langle 2\rangle$, $C_6\langle 2,3\rangle$, $C_7\langle 2,3\rangle$, and $C_8\langle 2,3,4\rangle$ do not appear in Table 1.

The first five circulants in Table 1 are matching unique by Theorem 2 above. In Farrell, Guo and Constantine [6], it was proven that $K_r \cup K_r \cup \ldots \cup K_r$ (n times) is matching unique for all positive integers n. Therefore $C_8\langle 2,4\rangle$ is matching unique. In [7], it was shown that if a graph is matching unique then so is its complement. Therefore, $\overline{B}_2 = C_8\langle 1,3,4\rangle$ is matching unique. Hence, all seven circulants (in Table 1) that are not chromatically unique are matching unique.

It is true that some of the 23 chromatically unique circulants are also matching unique. Therefore some circulants are both matching unique and chromatically unique. Hence, it is true that for all positive integers p less than nine, every circulant with p nodes is either matching unique or chromatically unique or both.

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