A Census of Dicyclic $(v, 4, 2)$-designs for $v \leq 22$

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1. Introduction

Hanani [3] was the first to determine necessary and sufficient conditions for the existence BIBDs with block size 4, i.e., (v, k, λ)-designs. In the particular case of $\lambda = 2$, the condition on the number of elements is especially simple: $v \equiv 1 \pmod{3}$.

The situation is far less satisfactory as far as enumeration of $(v, 4, \lambda)$-designs is concerned. The exact values of $N(v, 4, \lambda)$, the number of non-isomorphic $(v, 4, \lambda)$-designs, are known only for $v \leq 16$ if $\lambda = 1$, and for $v \leq 13$ if $\lambda = 2$ [2], [4]:

$$N(7,4,2) = 1, \quad N(10,4,2) = 3 \text{ [2]}, \quad N(13,4,2) = 2407 \text{ [4]}.$$

The current status of the existence problem for $(v, 4, \lambda)$-designs having additional properties, such as an automorphism of prescribed type, also leaves a lot to be desired. For instance, it has been conjectured, but remains far from proved that a cyclic $(v, 4, 1)$-design exists if and only if $v \equiv 1$ or $4 \pmod{12}$, $v \not\equiv 16, 25, 28$.

In this paper we take a look at the small dicyclic $(v, 4, 2)$-designs (for precise definitions, see Section 2). One of our objectives was to improve the lower bound

$N(22,4,2) \geq 1$ given in the tables [5]. Our main result is that there are exactly 7921 nonisomorphic dicylic $(22,4,2)$-designs. Of these designs, 43 have automorphism group of order 22 while the remaining 7878 have group of order 11.

Our other objective was to investigate the existence of $(v,4,2)$-designs admitting 2-resolutions. Here, a 2-resolution is a partition of the block set of the design into twofold parallel classes, or 2-PCs, where a 2-PC is a set of $v/2$ blocks with the property that each element is contained in exactly 2 blocks of it. Clearly, for a $(v,4,2)$-design to admit a 2-resolution, one must also have $v \equiv 0 \pmod 2$, thus $v \equiv 4 \pmod 6$. One of us (RCM) formulated the following problem: For which orders $v \equiv 4 \pmod 6$ does there exist a $(v,4,2)$-design admitting a 2-resolution? Although recently this question has been essentially settled [6], many interesting questions remain.

In addition to obtaining all nonisomorphic dicyclic $(22,4,2)$-designs, we investigated the 43 dicyclic designs with automorphicm group of order 22 for 2-resolutions. Each of the 40 cyclic designs admits a 2-resolution (some admit as many as 27896 distinct 2-resolutions!) but out of the three designs whose group is D_{11}, only one admits 2-resolutions.

2. Definitions

A balanced incomplete block design BIBD(v,b,r,k,λ) is a v-set V together with a collecton of b distinguished k-subsets of V called blocks such that each element of V is contained in exactly r blocks, and each 2-subset of V is contained in exactly λ blocks. Since $r = \lambda(v-1)/(k-1)$ and $b = \lambda v(v-1)/[k(k-1)]$, the two parameters r, b are usually supressed from notation, and, for brevity, we speak of a (v,k,λ)-design.

In this paper we are concerned with the case when $k = 4$ and $\lambda = 2$, i.e. with The case of $(v,4,2)$-designs. By using his powerful recursive method, Hanani [3] was the first to prove that a $(v,4,2)$-design exists if and only if $v \equiv 1 \pmod 3$. Let us mention in passing that till today no direct proof of this result is available (for more recent different recursive proofs, see [8], and [9]).

A (v,k,λ)-design is called cyclic if it has an automorphism permuting the elements in a single cycle of length v. It is called dicyclic if it has an automorphism consisting of two cycles of length $v/2$ each. Thus for a dicyclic $(v,4,2)$-design to exist, we must have $v \equiv 4 \pmod 6$. In the next two sections we enumerate the dicyclic $(16,4,2)$- and $(22,4,2)$-designs.

A q-fold parallel class (briefly, a q-PC) in a (v,k,λ)-design is a set R of qv/k blocks such that each element is contained in exactly q blocks of R. A q-resolution is a partition of the set of blocks into q-PCs. A (v,k,λ)-design admitting a q-resolution is called q-resolvable.

3. The dicyclic (16, 4, 2)-designs

Let the set of elements be $V = \{0,1,2,3,4,5,6,7,A,B,C,D,E,F,G,H\}$, and let $\alpha = (0\ 1\ 2\ 3\ 4\ 5\ 6\ 7)(A\ B\ C\ D\ E\ F\ G\ H)$ be the dicyclic automorphism of a (16, 4, 2)-design.

There exist two tactical configurations for the base blocks of a dicyclic (16, 4, 2)-design (here 1 and 2 stands for the element orbits of digits, and of letters, respectively):

Type I.	1112	1112	1222	1222	1122	
Type II.	1122	1122	1122	1122	1111*	2222*

(The asterisk indicates petite orbits of length two – the corresponding base blocks are 0246 and ACEG).

Moreover, solutions of type I split into two classes, depending on whether the orbit with the base block of type 1122 is a full orbit of length 8 (Type Ia) or a repeated short orbit of length 4 (type 1b).

The actual designs were generated by hand while the subsequent isomorph rejection was done by a computer using the general isomorphism program of Brendan McKay. It turns out that there are exactly 8 nonisomorphic dicyclic (16, 4, 2)-designs: four of type Ia (Nos. 1–4) and four of type Ib (Nos. 5–8) (any design of type II turns out to be isomorphic to one of type Ib). The base blocks for these 8 designs are listed in Table 1, together with the automorphism group order, block orbit partitioning, and the number of parallel classes. All 8 designs are transitive on elements but only one (No. 5, the "doubled affine plane") is block-transitive. Designs No. 1–4 have no repeated blocks, No. 5 has 20 repeated blocks, while Nos. 6–8 have 4 repeated blocks each. Designs No. 5,6,7 are decomposable, while Nos. 1,2,3,4 and 8 are indecomposable. All designs but No. 1 are resolvable.

Table 1

Design No.	Base Blocks					$\|G\|$	BOP	#PC	D	R
1	013E	014D	4ABD	3ABE	02AC	32	16 + 16 + 8	8	No	No
2	013E	014H	4ABD	7ABE	02AC	32	16 + 16 + 8	16	No	Yes
3	013C	014F	5ACD	5ABE	02AC	64	16 + 16 + 8	16	No	Yes
4	013G	014F	1ACD	5ABE	02AC	64	16 + 16 + 8	16	No	Yes
5	013C	013C	5ACD	5ACD	04AE	5760	40	80	Yes	Yes
6	013C	023B	6ABD	5ACD	04AE	192	32 + 8	32	Yes	Yes
7	013C	023F	2ABD	5ACD	04AE	768	32 + 8	48	Yes	Yes
8	012D	025H	2ABD	5ACD	04AE	256	32 + 8	32	No	Yes

$|G|$ = group order
#PC = number of parallel classes
BOP = block orbit partitioning
D = decomposable
R = resolvable

Table 2. Transitive (22,77,14,4,2)-designs

No.	Base blocks							2-PC	2-RES
1	0 1 3 4	0 2 12 16	0 4 20 21	0 5 11 19	0 5 12 20	0 11 13 15	0 13 18 19	279	0
2	0 1 3 4	0 2 19 20	0 4 13 17	0 5 12 19	0 5 15 16	0 11 14 16	0 12 15 21	532	1606
3	0 1 3 4	0 2 12 20	0 4 13 21	0 5 16 18	0 5 17 19	0 11 15 16	0 14 15 19	378	0
4	0 1 2 11	0 2 5 11	0 3 7 13	0 5 12 18	0 12 15 19	0 14 15 16	0 14 16 19	716	3366
5	0 1 2 11	0 2 5 11	0 3 7 15	0 5 12 18	0 13 17 21	0 14 15 16	0 14 16 19	441	55
6	0 1 2 11	0 2 5 15	0 3 7 12	0 5 13 19	0 11 15 18	0 12 14 17	0 16 17 18	793	27896
7	0 1 2 11	0 2 5 15	0 3 7 18	0 5 13 19	0 12 14 17	0 12 16 20	0 16 17 18	771	27896
8	0 1 2 11	0 2 5 17	0 3 7 14	0 5 13 19	0 12 15 21	0 13 16 20	0 16 17 18	716	7227
9	0 1 2 11	0 2 5 17	0 3 7 16	0 5 13 19	0 11 14 18	0 12 15 21	0 16 17 18	804	1892
10	0 1 2 11	0 2 5 17	0 3 7 19	0 5 13 18	0 11 14 20	0 14 18 21	0 15 16 17	507	143
11	0 1 2 11	0 2 5 17	0 3 7 21	0 5 13 18	0 11 14 20	0 12 16 19	0 15 16 17	562	9578
12	0 1 2 11	0 2 5 19	0 3 7 11	0 5 12 17	0 13 14 15	0 13 16 20	0 16 18 21	815	27896
13	0 1 2 11	0 2 5 19	0 3 7 16	0 5 12 17	0 11 15 19	0 13 14 15	0 16 18 21	859	27896
14	0 1 2 11	0 2 5 21	0 3 7 18	0 5 12 17	0 13 14 15	0 13 17 20	0 14 16 19	738	7227
15	0 1 2 11	0 2 5 21	0 3 7 20	0 5 12 17	0 11 15 18	0 13 14 15	0 14 16 19	716	1848
16	0 1 3 11	0 1 4 16	0 2 6 13	0 5 14 19	0 12 13 16	0 15 17 21	0 17 18 20	617	616
17	0 1 3 11	0 1 4 16	0 2 6 18	0 5 14 20	0 11 13 17	0 13 14 17	0 18 19 21	738	7337
18	0 1 3 11	0 1 4 17	0 2 6 16	0 5 12 18	0 11 15 20	0 12 19 20	0 14 15 17	617	4389
19	0 1 3 11	0 1 4 17	0 2 6 17	0 5 14 20	0 12 13 16	0 12 14 18	0 18 19 21	815	27896
20	0 1 3 11	0 1 4 17	0 2 6 18	0 5 15 20	0 11 19 20	0 12 14 18	0 13 14 17	606	814
21	0 1 3 11	0 1 4 20	0 2 6 16	0 5 11 17	0 12 13 15	0 13 18 20	0 14 15 18	606	308
22	0 1 3 11	0 1 4 21	0 2 6 11	0 5 12 17	0 13 14 16	0 13 15 19	0 14 15 18	584	726

Table 2. Transitive (22,77,14,4,2)-designs

No.	Base blocks							2-PC	2-RES
23	0 1 3 11	0 1 4 16	0 2 7 13	0 5 14 20	0 12 16 18	0 13 14 17	0 18 19 21	782	1892
24	0 1 3 11	0 1 4 16	0 2 7 17	0 5 14 19	0 11 13 18	0 12 13 16	0 17 18 20	496	913
25	0 1 3 11	0 1 4 17	0 2 7 11	0 5 12 18	0 12 19 20	0 14 15 17	0 14 16 21	650	880
26	0 1 3 11	0 1 4 17	0 2 7 14	0 5 14 20	0 11 15 17	0 12 13 16	0 18 19 21	859	27986
27	0 1 3 11	0 1 4 17	0 2 7 14	0 5 15 20	0 11 19 20	0 12 16 18	0 13 14 17	705	616
28	0 1 3 11	0 1 4 20	0 2 7 20	0 5 11 17	0 12 13 15	0 14 15 18	0 14 16 21	540	11
29	0 1 3 11	0 1 4 21	0 2 7 15	0 5 12 17	0 11 16 20	0 13 14 16	0 14 15 18	727	528
30	0 1 3 11	0 1 8 20	0 2 6 16	0 5 12 18	0 11 15 20	0 13 16 17	0 14 15 17	562	880
31	0 1 3 11	0 1 8 13	0 2 6 13	0 5 14 19	0 12 15 16	0 15 17 21	0 17 18 20	837	638
32	0 1 3 11	0 1 8 13	0 2 6 17	0 5 14 20	0 12 14 18	0 13 16 17	0 18 19 21	881	27896
33	0 1 3 11	0 1 8 14	0 2 6 18	0 5 14 20	0 11 13 17	0 12 15 16	0 18 19 21	804	1892
34	0 1 3 11	0 1 8 14	0 2 6 18	0 5 15 20	0 11 19 20	0 12 14 18	0 13 16 17	540	913
35	0 1 3 11	0 1 8 15	0 2 6 11	0 5 12 17	0 13 14 16	0 13 15 19	0 17 20 21	683	682
36	0 1 3 11	0 1 8 15	0 2 6 16	0 5 11 17	0 12 13 15	0 13 18 20	0 16 19 20	584	308
37	0 1 3 11	0 1 8 20	0 2 7 11	0 5 12 18	0 13 16 17	0 14 15 17	0 14 16 21	661	4389
38	0 1 3 11	0 1 8 13	0 2 7 14	0 5 14 20	0 11 15 17	0 13 16 17	0 18 19 21	881	27896
39	0 1 3 11	0 1 8 13	0 2 7 17	0 5 14 19	0 11 13 18	0 12 15 16	0 17 18 20	672	726
40	0 1 3 11	0 1 8 14	0 2 7 13	0 5 14 20	0 12 15 16	0 12 16 18	0 18 19 21	672	7227
41	0 1 3 11	0 1 8 14	0 2 7 14	0 5 15 20	0 11 19 20	0 12 16 18	0 13 16 17	617	528
42	0 1 3 11	0 1 8 15	0 2 7 20	0 5 11 17	0 12 13 15	0 14 16 21	0 16 19 20	540	11
43	0 1 3 11	0 1 8 15	0 2 7 15	0 5 12 17	0 11 16 20	0 13 14 16	0 17 20 21	540	913

4. The dicyclic $(22,4,2)$-designs

Let $V = \{0,1,\ldots,21\}$, and let $\alpha = (0\ 1\ \ldots\ 10)(11\ 12\ \ldots\ 21)$ be the dicyclic automorphism of a $(22,4,2)$-design. We have $b = 77$, and so necessarily the blocks of any such design will fall into 7 full length (i.e. length 11) orbits under α. There are essentially two different tactical configurations for the 7 base blocks (again, 1 and 2 stand for the two element-orbits):

I.	1111	1122	1122	1122	1122	1222	1222
II.	1112	1112	1112	1122	1222	1222	1222.

(Another tactical configuration is obtained from that of type I by interchanging 1 and 2.) For reasons that are self-evident, we refer to these two configuratons (and the respective solutions) as asymmetric, and symmetric, respectively.

The designs of both types were generated on a Sun-4 computer by a lexicographic hierarchical complete backtrack from all multiplier non-equivalent starts corresponding to the two tactical configurations. The programs were written in the C programming language. The isomorph rejection was greatly assisted by a straightforward extension of the Bays-Lambossy multiplier theorem [1] (see also [7]), according to which, if two dicyclic $(22,4,2)$-designs are isomorphic then there exists a multiplier isomorphism preserving orbits in the asymmetric case, and preserving or interchanging orbits in the symmetric case, respectively.

Nevertheless, after obtaining what we believed to be a complete set of nonisomorphic dicyclic designs, the general purpose isomorphism testing program of Brendan McKay was used to check this independently.

The number of nonisomorphic dicyclic $(22,4,2)$-designs turns out to be surprisingly large. There are 4546 nonisomorphic designs of the asymmetric type, and 3375 designs of the symmetric type, for a total of 7921 nonisomorphic dicyclic $(22,4,2)$-designs. This number is obviously too large for a complete analysis.

A reasonable size subclass is provided by the 43 designs having automorphism group of order 22 (all the other 7878 designs have group of order 11). Of these 43 designs, three are of asymmetric type and have as their automorphism group the dihedral group D_{11}. The other 40 designs of symmetric type are all cyclic. The base blocks for these 43 designs appear in Table 2. For each of these designs, all twofold parallel classes and all 2-resolutions were generated; their numbers are also given in Table 2. Each of the 40 cyclic designs is 2-resolvable, with the number of distinct 2-resolutions ranging from 11 to 27896. Interestingly, only one of the three designs with the dihedral group (design No. 2) is 2-resolvable. An example of a 2-resolution of the design No. 2 is given in Table 3.

Table 3.

A 2-resolution of the design No. 2.

2-PC	Twofold	parallel	classes			
1	0 1 3 4	0 12 15 21	1 11 13 16	2 7 17 18	2 8 12 13	3 14 17 19
	4 15 18 20	5 6 8 9	5 16 19 21	6 7 9 10	10 11 14 20	
2	0 7 8 10	0 11 14 16	1 2 4 5	1 12 15 17	2 3 5 6	3 9 13 14
	4 9 19 20	6 16 18 21	7 11 17 19	8 12 18 20	10 13 15 21	
3	0 1 8 9	0 5 15 16	1 2 9 10	2 12 14 17	3 4 6 7	3 13 15 18
	4 14 16 19	5 10 20 21	6 11 17 20	7 12 18 21	8 11 13 19	
4	0 2 3 10	0 6 11 21	1 6 16 17	1 7 11 12	2 13 16 18	3 8 18 19
	4 5 7 8	4 10 14 15	5 15 17 20	9 12 14 20	9 13 19 21	

5	i i+2 j+19 j+20	$i, j \in Z_{11}$; first two elements reduced to the range $\{0, 1, \ldots, 10\}$, the last two to $\{11, 12, \ldots, 21\}$
6	i i+4 j+13 j+17	
7	i i+5 j+12 j+19	

5. Conclusion

As a consequence of the results of Section 4, the bound for the number of nonisomorphic designs with parameter set No.100 in the tables [5] can now be improved to $Nd(22, 77, 14, 4, 2) \geq 7921$.

It appears reasonable to conjecture that a dicyclic $(v, 4, 2)$-design exists if and only if $v \equiv 4 \pmod 6$ (of the three nonisomorphic $(10, 4, 2)$-designs, cf. [2], exactly one is dicyclic). We also conjecture that a cyclic $(v, 4, 2)$-design exists if and only if $v \equiv 1 \pmod 3$, $v \not\equiv 10$.

Several questions can be asked concerning 2-resolutions of $(v, 4, 2)$-designs. For example, is it true that for every $v \equiv 4 \pmod 6$ there exists a $(v, 4, 2)$-design *without* a 2-resolution? If yes, what is the smallest q such that every such design admits a q-resolution? The computational results of this paper leave lots of room for speculation.

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