A Note On The Characterization Of Thistles By Their Matching Polynomials

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Abstract. It is also shown that for a certain family of graphs (called thistles), the coefficients of the matching polynomial repeat themselves symmetrically. This turns out to be a characterizing property for some thistles.

1. Introduction.

First of all, we will give some definitions which are crucial to the material which follows.

Let (G, u) and (H, v) be two graphs, rooted at u and v respectively. The *coalescence* of G and H, is the graph obtained by identifying the roots u and v. We simply say that H is *attached* to G (or G is attached to H), if the roots are clear or unimportant. The nodes u and v are called *nodes of attachment*. We will denote by $G(H)$, the regular graph obtained from G and the rooted graph (H, u), by simultaneous coalescence of a copy of H, at u, to every node of G. We then say that H is attached to every node of G. In the special case when H is an edge (twig), $G(H)$ will be called a *thistle*.

Let G be a graph consisting of a graph G_0 with non-trivial graphs (graphs with more than one node) attached to it. Then G_0 is called the *core* of G, if G_0 itself is not formed by non-trivial attachments (that is, attachments of graphs with more than one node.)

We show that for all thistles the coefficients of the matching polynomial repeat themselves symmetrically and that this property characterizes certain thistles. Our results generalize a result given in Godsil and McKay [5].

2. Simple thistles.

Let G_0 be a core. We can define a recursive family of regular cacti as follows. The first member is obtained by attaching the rooted graph (H, u) to every node of G_0, that is, $G_1 = G_0(H)$. The second member is $G_2 = G_1(H)$, etc. The kth member is $G_k = G_{k-1}(H)$.

Definition: A family is called *simple* if G_0 is a node. A simple family is interesting. In this case, $G_1 = G_0(H) = H$, so the first member is the attached graph. We can also observe that H is simply H itself, with nodes attached, that is. Therefore $G_1 = G_0(H) = H(G_0)$.

$$G_2 = G_1(H) = H(H) = H(G_1)$$
$$G_3 = G_2(H) = H(G_1)(H).$$

Now, the attachment of H to every node of an attached graph G_1 yields $G_1(G_1) = G_2$. Hence we get $G_3 = G_1(G_2) \cdot \Rightarrow G_3 = G_2(G_1) = G_1(G_2)$.

It is clear that the above argument can be repeated for G_4, G_5, etc. Hence we have the following theorem which can be formally proved by induction on k.

Lemma 1. *For a simple family,*

$$G_k = G_{k-1}(G_1) = G_1(G_{k-1}).$$

We can use the thistle function (see Farrell [2]) and Lemma 1 in order to obtain a formula for the matching polynomial of G_k.

It has been observed that in the matching polynomials of simple thistles; and indeed in all thistles, the coefficients repeat themselves symmetrically. We will prove a theorem which formalizes this observation.

Lemma 2. *Let (G_r) be a family of thistles with core H, where $|V(H)| = p$ and $|E(H)| = q$, then G_r has $2^r p$ nodes, and $q + 2^r p$ edges.*

Proof: The result can be easily established by induction on r. ∎

From the above theorem, we see that G_r must have an even number of nodes, since $2^r p$ is even, for $r > 0$.

Theorem 1. *Let G_k be the kth descendant of a family of thistles. Let*

$$M(G_k;\underline{w}) = \sum_{r=0}^{p} a_r w_1^{2p-2r} w_2^r,$$

where $2p$ is the number of nodes in G_k. Then

$$a_r = a_{p-r}, \text{ for } 0 \le r \le p$$

(that is, $M(G_k;\underline{w})$ is symmetric).

Proof: Let

$$M(G_{k-1};\underline{w}) = \sum_{i=0}^{s} a_i w_1^{2s-2i} w_2^i,$$

where $2s$ in the number of nodes in G_{k-1}. Since G_k is obtained from G_{k-1} by adding twigs, $G_k = G_{k-1}(H)$, where H is a twig. It follows that by using the thistle function for matching polynomials, we get

$$\begin{aligned} M(G_k; \underline{w}) &= \sum_{i=0}^{s} a_i \left(w_1^2 + w_2\right)^{2s-2i} \left(w_1^2 w_2\right)^i \\ &= \sum_{i=0}^{s} a_i \sum_{j=0}^{2s-2i} \binom{2s-2i}{j} w_1^{2j} w_2^{2s-2i-j} w_1^{2i} w_2^{i} \\ &= \sum_{i=0}^{s} \sum_{j=0}^{2s-2i} a_i \binom{2s-2i}{j} w_1^{2i+2j} w_2^{2s-i-j}. \end{aligned}$$

From this, we get that the coefficient of w_2^r is

$$\sum_{i=0}^{s} a_i \binom{2s-2i}{2s-r-i} = \sum_{i=0}^{s} a_i \binom{2s-2i}{r-i}.$$

G_k will have $4s$ nodes since G_{k-1} has $2s$ nodes. Therefore $p = 2s$.

$$\begin{aligned} \text{Coefficient of } w_2^{2s-r} &= \sum_{i=0}^{s} a_i \binom{2s-2i}{2s-(2s-r)-i} \\ &= \sum_{i=0}^{s} a_i \binom{2s-2i}{r-i} \\ &= \text{coefficient of } w_2^r. \end{aligned}$$

∎

Theorem 1 is interesting. It establishes the existence of a general class of graphs whose matching polynomials are perfectly symmetric. As far as we know, thistles are the first general class of graphs that have been identified as having this property. The following theorem shows that this properly characterizes thistles which have no triangles, that is, *triangle-free* thistles.

Theorem 2. *Let G be a triangle-free connected graph. Then G is a thistle if and only if its matching polynomial is symmetric.*

Proof: The necessary part of the theorem has already been established in Theorem 1. We will now show that if the matching polynomial of a graph G is symmetric then G must be a thistle.

Let $M(G; \underline{w}) = \sum_{r=0}^{[\frac{p}{2}]} a_r w_1^{p-2r} w_2^r$; and assume that $a_r = a_{[\frac{p}{2}]-r}$, for $0 \leq r \leq [p/2]$. Clearly $a_0 = 1$ and $a_1 = q$, the number of edges in G. Therefore $a_{[p/2]} = 1$ and $a_{[p/2]-1} = q$. It follows that G has exactly one perfect matching and q defect-2 matchings (matchings with 2 nodes). Since G has a perfect matching, it must have an even number of nodes $\Rightarrow p = 2s$, for some positive integer s. Therefore, the perfect matching C in G will contain s edges.

Let us count the defect-2 matchings in G. We can omit any edge of C and obtain a defect-2 matching. Thus, there are s of these. For every edge ab of G which does not belong to C, ax and by are edges in C, for some nodes x and y in G. By using the edge ab and keeping nodes x and y as isolated nodes, we can obtain a defect-2 matching by including all the other edges of C. There will be $q - s$ defect-2 matchings of this type. Since each of these matchings contains an edge that does not belong to C, they will all be distinct from those already counted above. Hence, up to this point we have $s + q - s = q$ defect-2 matchings. Therefore, there are no other defect-2 matchings in G.

Suppose that G contains a node x such that $d(x)$ (the valency of x) > 1 and x is not adjacent to any node of valency 1. Since G has a perfect matching, $d(x) \neq 0$. Therefore $d(x) \geq 2$. Thus $\exists$ nodes y and z adjacent to x, such that $d(y) > 1$ and $d(z) > 1$ and $yz \notin E(G)$ (since G is triangle free). Without loss in generality, we will assume that the edge xz belongs to C. Then $\exists$ nodes t, u and v such that t and u are adjacent to y and z respectively; u is adjacent to v and ty and uv belong to C. Consider the path $tyxzuv$. We can cover this path, by using the edges xy and zu, together with the isolated nodes t and v. This cover, together with the remaining edges of C, yield a new (since it contains 2 edges of G that do not belong to C) defect-2 matching in G. This is a contradiction. Therefore, no such node x exist. Hence every node in G is either of valency 1 or else is adjacent to a node of valency 1.

Clearly no node of G can be adjacent to more than one node of valency 1, since G has a perfect matching. Therefore, every node of G is either of valency 1 or else is adjacent to exactly one node of valency 1. It follows that G is a thistle. This completes the proof. ∎

It has been shown (see Farrell [3]) that the circuit polynomial of a graph G (denoted by $C(G; \underline{w})$) is related to the characteristic polynomial of G (denoted by $\phi(G; \lambda)$), as follows;

$$\phi(G; \lambda) = C(G; (\lambda, -1, -2, -2, \ldots)).$$

However, if G is a tree, then its only circuit covers are matchings. Therefore $M(G; \underline{w})$ and $C(G; \underline{w})$ coincide, for trees. Hence, if T is a tree,

$$\phi(T; \lambda) = M(T; (\lambda, -1)),$$

Clearly, a thistle is a tree if and only if its core is a tree. Hence, we have the following corollary.

Corollary 2.1. *Let T be a tree. Then T is a thistle if and only if its characteristic polynomial is symmetric.*

Corollary 2.1 is essentially the result given in Godsil and McKay [5] (Theorem 3.2). Hence, Theorem 2 is a generalization of their result. It should also be noted that all the results given in this paper also hold for characteristic polynomials, if we restrict the graphs to be trees.

5. Discussion.

It has been observed by several authors, that the coefficients of the matching polynomial increase then decrease. This of course, has also been observed for other graph polynomials, including chromatic polynomials. Theorem 2 has identified a class of graphs for which the distribution of the coefficients is symmetrical about the largest value. Since thistles have an even number of nodes and also a perfect matching, it follows that their matching polynomials have an odd number of terms. Thus, the symmetry is quite a "beautiful" one. As the theorem implies, no other triangle free graphs have this property.

References

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