ACCURATE BOUNDS FOR NEW VAN DER WAERDEN TYPE NUMBERS

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Abstract. Numbers similar to those of van der Waerden are examined by considering sequences of positive integers $\{x_1, x_2, \ldots, x_n\}$ with $x_{i+1} = x_i + d + r_i$, where $d \in Z^+$ and $0 \le r_i \le \max(0, f(i))$ for a given function f defined on Z^+. Let $w_f(n)$ denote the least positive integer such that if $\{1, 2, \ldots, w_f(n)\}$ is 2-colored, then there exists a monochromatic sequence of the type just described. Tables are given of $w_f(n)$ where $f(i) = i - k$ for various constants k, and also where $f(i) = i$ if $i \ge 2$, $f(1) = 0$. In this latter case, as well as for $f(i) = i$, an upper bound is given that is very close to the actual values. A tight lower bound and fairly reasonable upper bound are given in the case $f(i) = i - 1$.

1. Introduction.

In 1927 van der Waerden [8] proved the existence of numbers $w(n)$, defined to be the least positive integer which guarantees that if $\{1, 2, \ldots, w(n)\}$ is partitioned into two sets, then one of the two sets contains an arithmetic progression of length n. The only known nontrivial values of $w(n)$ are $w(3) = 9$, $w(4) = 35$, and $w(5) = 178$ (see [2] and [7]). Until very recently, all known proofs of van der Waerden's theorem yielded such weak upper bounds on $w(n)$ that they were not even primitive recursive functions of n. Shelah [6] has recently obtained a primitive recursive upper bound, although it is rather large. For example, it is still unknown if $w(n)$ is bounded above by a tower of n 2's.

In [3], [4], and [5], Landman and Greenwell found reasonable upper bounds, as well as exact values, for numbers analogous to $w(n)$ by considering a larger class of sequences than the class of arithmetic progressions, namely those generated by iteration of a polynomial function with integer coefficients. In this paper we examine a different generalization of arithmetic sequences, which can loosely be described as arithmetic sequences with some slack allowed. The significance of this class of sequences is that we are able to prove upper bounds for the corresponding Ramsey numbers which, unlike those in [3–5], are quite close to the actual values. Further, by sufficiently decreasing the allowable slack, these numbers eventually become the van der Waerden numbers themselves.

The following terminology and notation will be used throughout the paper. We denote $\{1,2,\ldots,k\}$ by $[1,k]$. A 2-coloring of $[1,k]$ refers to a function $\chi: [1,k] \rightarrow \{0,1\}$. A set X is monochromatic under a coloring χ if χ is constant on X. If f is a function defined on Z^+, then a w_f-sequence of length n is a sequence of positive integers $\{x_1,x_2,\ldots,x_n\}$ with $x_{i+1} = x_i + d + r_i$ for $i = 1,\ldots,n-1$ where $d, r_i \in Z^+$ and $0 \leq r_i \leq \max(0,f(i))$. A sequence is a $\overline{w}$-sequence if $f(i) = i$ for $i \geq 2$ and $f(1) = 0$. The symbol $\alpha(n)$ will denote the least positive integer which guarantees that if $[1,\alpha(n)]$ is 2-colored, then there exists a monochromatic α-sequence of length n, where α-sequence refers to any type of sequence under consideration. Note that $w_{f(i)}(n) \geq w_{g(i)}(n)$ if $f(i) \leq g(i)$ for $i = 1,\ldots,n-1$, and $w_i(n) \leq \overline{w}(n) \leq w_{i-1}(n)$. Note also that $w_{i-k}(n) = w(n)$ for $k \geq n-1$.

2. Bounds.

We have an upper bound for $w_i(n)$ and a lower bound for $w_{i-1}(n)$, both of which are almost identical to the values of the functions given in the next section. We also have an upper bound for $\overline{w}(n)$ which is very close to the known values. Finally, we have an upper bound for $w_{i-1}(n)$ which is not too much larger than the true values.

Theorem 1. *Let $n \geq 2$. If $[1,n(n+1)/2]$ is 2-colored, then there exists a monochromatic w_i-sequence of length n with $d=1$. Hence $w_i(n) \leq n(n+1)/2$.*

We omit the proof of Theorem 1, since it is similar to but slightly simpler than the proof of Theorem 2.

Theorem 2. *For $n \geq 3$, if $[1,n-2+n(n+1)/2]$ is 2-colored, then there exists a monochromatic $\overline{w}$-sequence $\{x_1,\ldots,x_n\}$ with $d \leq 2$ and $x_n + d \leq n-1+n(n+1)/2$. Hence $\overline{w}(n) \leq (n-1)(n+4)/2$.*

Proof: An examination of the 2-colorings of [1, 7] shows that the theorem is true for $n = 3$. We proceed by induction on n. Assume that the result holds for n and 2-color $[1,n-1+\sum_{i=1}^{n+1} i] = A$. By the induction hypothesis there exists a monochromatic $\overline{w}$-sequence $\{x_1,\ldots,x_n\}$ with $d \leq 2$ and $x_n+d \leq n-1+\sum_{i=1}^{n} i$. Suppose it is colored red. Let $X=\{x_n+d, x_n+d+1,\ldots,x_n+d+n\} \subseteq A$. If X is colored blue, then X is a monochromatic $\overline{w}$-sequence of length $n+1$ contained in A with $d' = 1$, so that $x_n + d + n + d' \leq n + \sum_{i=1}^{n+1} i$. Otherwise, if $x_n + d + j$ is colored red for some j, then $\{x_1,\ldots,x_n,x_n+d+j\}$ is a red $\overline{w}$-sequence with $d \leq 2$ satisfying $x_n + d + j + d \leq n + \sum_{i=1}^{n+1} i$. Thus in either case the theorem holds. ∎

Theorem 3. $w_{i-1}(n) \geq 2(n-1)^2 + 1$.

Proof: For $j = 0,1,\ldots,2n-3$, define $\chi: [1,2(n-1)^2] \rightarrow \{0,1\}$ by $\chi(A_j) = 0$ if j is even and $\chi(A_j) = 1$ if j is odd, where $A_j = \{j(n-1)+1, j(n-1)+2,$

$\dots, (j+1)(n-1)\}$. Assume $\{x_1, \dots, x_n\}$ is a monochromatic w_{i-1}-sequence under χ. We show that all the x_i must be in the same A_j, which contradicts the size of A_j.

Suppose $x_1 \in A_j$. If $x_2 \notin A_j$, then $x_2 \in A_s$ where $s \geq j+2$, so $x_2 - x_1 \geq n$. Then since $x_{i+1} - x_i \geq x_2 - x_1$, and since $|A_k| = n = 1$ for each k, all the x_i's would have to be in distinct A_k's, but there are only $n-1$ A_k's for which k has the same parity as j. Hence x_2 must be in A_j. Assume that $2 \leq k \leq n-1$ and that $x_1, x_2, \dots, x_k \in A_j$. If $x_{k+1} \notin A_j$, then $x_{k+1} - x_k \geq n$, and since $r_k \leq k-1$, we have $d \geq n-(k-1)$. However, since $|A_j| = n-1$, we have $d \leq (n-2)/(k-1) \leq n-k$, a contradiction. Thus $x_{k+1} \in A_j$. By induction, we have $x_1, \dots, x_n \in A_j$, completing the proof. ∎

Theorem 4. *For* $n \geq 2$, $w_{i-1}(n) \leq m_n = 1 + 2(n-1)^2 + 2\sum_{i=4}^{n-1} i(i-1)$, *and every 2-coloring of* $[1, m_n]$ *contains a monochromatic* n*-term* w_{i-1}*-sequence with* $d \leq 2(n-1)$.

Proof: Since $w_{i-1}(2) = 3$, $w_{i-1}(3) = 9$, and $w_{i-1}(4) = 19$, the result clearly holds for $n = 2, 3$, and 4. We proceed by induction on n. Assume the result holds for $n \geq 4$ and 2-color $[1, m_{n+1}]$ with χ. Thus there exists a monochromatic w_{i-1}-sequence $X = \{x_1, \dots, x_n\}$ with $d \leq 2(n-1)$. Let $\chi(X) = 1$. For $i = 0, \dots, 2n$, define $A_i = \{x_n + d + ni, x_n + d + ni + 1, \dots, x_n + d + ni + (n-1)\}$. Note that $x_n + d + 2n^2 \leq 1 + 2(n-1)^2 + 2\sum_{i=4}^{n-1} i(i-1) + 2(n-1) + 2n^2$ $= 1 + 2n^2 + 2\sum_{i=4}^{n} i(i-1) = m_{n+1}$. If there were no $(n+1)$-term monochromatic w_{i-1}-sequence Y in $[1, m_{n+1}]$ with $d(Y) \leq 2n$, then all members of A_0 must have color 0. Hence, all members of A_i must have color 1 for i odd and color 0 for i even. But then $Y = \{x_n + d + ni : i = 0, \dots, n\}$ is a monochromatic w_{i-1}-sequence with $d(Y) = 2n$, so the theorem follows. ∎

We note that we have a degree 2 polynomial upper bound for $w_i(n)$, and a degree 3 polynomial upper bound for $w_{i-1}(n)$. We conjecture that there is a degree $k+2$ polynomial upper bound for $w_{i-k}(n)$.

3. Numerical values.

Using the algorithm described in [1] and [5], we have computed the following values on an IBM-PC using Turbo-Pascal, except for $w_{i-4}(5)$, which we took from [7].

$\backslash n$	3	4	5	6	7	8	9	10
w_i	6	10	14	20	27	34	44	52
$\overline{w}$	7	11	16	21	29	36	45	55
w_{i-1}	9	19	33	52	74	100		
w_{i-2}	**9**	22	38	60				
w_{i-3}	...	**35**	59	$\geq$88				
w_{i-4}	...	...	**178**					

Since $w_{i-k}(n) = w(n)$ for $k \geq n - 1$, one approaches the van der Waerden numbers, which are printed in bold face, by moving down the columns. Filling out more values in the table could give clues as to the size of other van der Waerden numbers, but the computation time became excessive. For example, $w_{i-3}(5)$ took 5 days and $w_{i-2}(6)$ took 27 days.

References

1. M. D. Beeler and P. E. O'Neil, *Some new van der Waerden numbers*, Discrete Mathematics **28** (1979), 135–146.
2. V. Chvátal, *Some unknown van der Waerden numbers*, in "Combinatorial Structures and Their Applications", R. K. Guy *et al*, eds., Gordon and Breach, New York, 1970.
3. R. N. Greenwell and B. M. Landman, *On the existence of a reasonable upper bound for the van der Waerden numbers*, Journal of Combinatorial Theory Series A **50** (1989), 82–86.
4. B. M. Landman, *Generalized van der Waerden numbers*, Graphs and Combinatorics **2** (1986), 351–356.
5. B. M. Landman and R. N. Greenwell, *Values and bounds for Ramsey numbers associated with polynomial iteration*, Discrete Mathematics **68** (1988), 77–83.
6. S. Shelah, *Primitive recursive bounds for van der Waerden numbers*, Journal of the American Mathematical Society **1** (1988), 683–697.
7. R. S. Stevens and R. Shantaram, *Computer generated van der Waerden partitions*, Mathematics of Computation **17** (1978), 635–636.
8. B. L. van der Waerden, *Beweis einer Baudetschen Vermutang*, Nieuw Arch. Wisk. **15** (1927), 212–216.