ON THE SET $J_0(v)$ FOR STEINER QUADRUPLE SYSTEMS

Gaetano Quattrocchi

Dipartimento di Matematica
Università di Catania
Viale A. Doria 6
95125 Catania
ITALY

Abstract. We prove that for every $v \equiv 4,8 \pmod{12}$ with $v \geq 16$, there exists a pair of $S(3,4,v)$s having exactly $k \in \{0,1,\dots,\lfloor\frac{v}{4}\rfloor\}$ pairwise disjoint blocks in common.

A Steiner quadruple system is a pair (V,B) where V is a finite set of size v (called the *order* of the system) and B is a collection of 4-subsets of V (called *blocks* or *quadruples*) such that every 3-subset of V is contained in exactly one block of B.

It is well-known [2] that an $S(3,4,v)$ exists if and only if $v \equiv 2,4 \pmod 6$.

The following intersection problem was posed by Micale [8]: Determine the set $J_0(v)$ of all integers k such that there exists a pair of $S(3,4,v)$s having exactly k pairwise disjoint blocks in common.

Let $I_0(v) = \{0,1,\dots,\lfloor\frac{v}{4}\rfloor\}$ ($\lfloor\frac{v}{4}\rfloor$ denotes the maximum integer $\leq \frac{v}{4}$), $v \equiv 2,4 \pmod 6$. In [9] it is poved that $J_0(v) = I_0(v)$ for every $v = m\cdot 2^n$ with $n \geq 2$ and $m = 4,5,7$; $J_0(4) = 1$, $J_0(8) = \{0,2\}$, $J_0(10) = \{0\}$ [4], and $\{0,1,2\} \subseteq J_0(14) \subseteq \{0,1,2,3\}$.

The object of this note is to prove that $J_0(v) = I_0(v)$ for every $v \equiv 4,8 \pmod{12}$, $v \geq 16$.

The reader can see [9] for an up-to-date survey on Steiner systems intersecting in a set with additional properties.

Let (V,B) be an $S(3,4,v)$. We will denote by V' the finite set such that $x' \in V'$ if and only if $x \in V$. Let B' be the collection of blocks such that $\{x',y',a',b'\} \in B'$ if and only if $\{x,y,a,b\} \in B$. Let $F = \{F_1,F_2,\dots,F_{v-1}\}$ be a 1-factorization of the complete graph K_v on the set V; also we will denote by $F' = \{F'_1,F'_2,\dots,F'_{v-1}\}$ the 1-factorization on V' such that $\{x',y'\} \in F'_i$ if and only if $\{x,y\} \in F_i$.

Let x be a fixed point of V. For every $y \in V$, $y \neq x$, define $F_y = \{\{a,b\}/\{x,y,a,b\} \in B\} \cup \{\{x,y\}\}$. Clearly $F(V,B,x) = \{F_y/y \in V - \{x\}\}$ is a 1-factorization of V (called a *Steiner 1-factorization* [7]).

We now describe two well-known constructions for quadruple systems.

Research supported by GNSAGA of CNR.

Construction A (e.g. see [6]). Let (X, A) and (X', B) be two $S(3,4,v)s$. Let $F = \{F_1, F_2, \ldots, F_{v-1}\}$ and $G = \{G_1, G_2, \ldots, G_{v-1}\}$ be any two 1-factorizations of X. Define a collection E of blocks on the set $S = X \cup X'$, as follows:

(a_1) Any block belonging to A or B belongs to E; and

(a_2) if $x_1, x_2 \in X$ ($x_1 \neq x_2$) and $y_1', y_2' \in X'$ ($y_1' \neq y_2'$) then $\{x_1, x_2, y_1', y_2'\} \in E$ if and only if $\{x_1, x_2\} \in F_i$ and $\{y_1', y_2'\} \in G_i'$.

Obviously (S, E) is an $S(3,4,2v)$. We will denote (S, E) by $[X \cup X'][A, B, F, G']$.

Construction B (see [1, 5]). Let (Q, C) be an $S(3,4,v)$, and let α be a permutation of Q. Define a collection D of blocks on the set $P = Q \cup Q'$ as follows:

(b_1) For every block $\{x, y, a, b\} \in C$ construct the following 8 blocks and place them in the set D: $\{\{x, y, a, (b\alpha)'\}, \{x, y, (a\alpha)', b\}, \{x, (y\alpha)', a, b\}, \{(x\alpha)', y, a, b\}, \{x', y', a', b\alpha^{-1}\}, \{x', y', a\alpha^{-1}, b'\}, \{x', y\alpha^{-1}, a', b'\}, \{x\alpha^{-1}, y', a', b'\}\}$.

(b_2) For every $x, y \in Q$, $x \neq y$, let $\{x, y, (x\alpha)', (y\alpha)'\} \in D$. Denote (P, D) (which is clearly an $S(3,4,v)$) by $((Q \cup Q'), C, \alpha)$.

Lemma 1. *For every* $u \equiv 2$ *or* $4 \pmod 6$ *with* $u \geq 2$, *we have* $I_0(2u) \backslash \{u/2 - 1\} \subset J_0(2u)$.

Proof: Let (Q, C) be an $S(3,4,u)$, let i be the identity permutation on Q, and let α_n be a permutation on Q with no fixed points and exactly n 2-cycles in its disjoint cycle decomposition. The permutation is easily constructed for any $n \in \{0, 1, 2, \ldots, u/2\} \backslash \{u/2 - 1\}$. Now the systems $((Q \cup Q'), C, i)$, and $((Q \cup Q'), C, \alpha_n)$ intersect in precisely n mutually disjoint blocks, thus proving the Lemma. ∎

Lemma 2. *If there exists an* $S(3,4,v)$ *with a subsystem of order* u, *and* $k \in J_0(2u)$ *then* $\{k, k + (v-u)/2\} \subset J_0(2v)$.

Proof: Let $Q = \{0, 1, \ldots, v-1\}$ and $P = \{0, 1, \ldots, u-1\}$. Consider the two Steiner 1-factorizations $F = F(Q, C, 0) = \{F_1, F_2, \ldots, F_{v-1}\}$ and $G = F(P, A, 0) = \{G_1, G_2, \ldots, G_{u-1}\}$. Clearly $G_i \subset F_i$ for $i = 1, 2, \ldots, u-1$. Let $\overline{F}_i = F_{\alpha(i)}$ and $\overline{G}_i = G_{\alpha(i)}$, $\alpha = (1\ 2\ \ldots\ u-1)\ (u\ u+1\ \ldots\ v-1)$. Then $(S, E_1) = [Q \cup Q'][C, C', F, \overline{F}']$ and $(S, D_1) = ((Q \cup Q'), S)$ contain the sub-systems $(R, E_2) = [P \cup P']\ [A, A', G, \overline{G}']$ and $(R, D_2) = ((P \cup P'), A)$ respectively. If (R, B_1) and (R, B_2) are two $S(3,4,2u)s$ such that $k \in J_0(2u)$, then $(S, (E_1 - E_2) \cup B_1)$ and $(S, (D_1 - D_2) \cup B_2)$ are two $S(3,4,2v)$ such that $k \in J_0(2v)$. To show that $k + (v-u)/2 \in J_0(2v)$, use the permutation $\alpha = (2\ 3\ \ldots\ u-1)(u\ u+1\ \ldots\ v-1)$. ∎

Theorem. $J_0(v) = I_0(v)$ *for* $v \equiv 4, 8 \pmod{12}$, $v \geq 16$.

Proof: By Lemma 1 it only remains to show that $v/4 - 1 \in J_0(v)$ for all $v \equiv 4$ or $8 \pmod{12}$ with $v \geq 16$. Now, since there exists an $S(3,4,v)$ with a subsystem of order 8 for all $v \equiv 2$ or 4 (mod 6) with $v \geq 16$ [3], and $3 \in J_0(16)$ [8] w have $3 + (v - 8)/2 = 2v/4 - 1 \in J_0(2v)$ for all $v \equiv 2$ or 4 (mod 6) with $v \geq 16$. Micale's results for $v = 16, 20$ and 28 complete the proof of the theorem. ∎

Acknowledgements.

The author is highly grateful to the referee, whose suggestions have significantly simplified the proof of the result.

References

1. J. Doyen and M. Vandensavel, *Non-isomorphic Steiner quadruple systems*, Bull. Soc.Math. Belg. **17** (1981), 17–26.
2. H. Hanani, *On quadruple systems*, Canad.J.Math. **12** (1960), 145–157.
3. A. Hartman, *Quadruple systems containing* $AG(3,2)$, Discrete Math. **39** (1982), 293–299.
4. E.S. Kramer and D.M. Mesner, *Intersections among Steiner systems* J. Combin. Theory (A) **16** (1974), 273–285.
5. C.C. Lindner, *A note on disjoint Steiner quadruple systems*, Ars Combinatoria 3 (1977), 271–276.
6. C.C. Lindner and A.Rosa, *Steiner quadruple systems – a survey*, Discrete Math. **21** (1978), 147 – 181.
7. E. Mendelsohn and A. Rosa, *One-factorizations of the complete graph – a survey*, J. Graph Theory **9** (1985), 43–65.
8. B. Micale, *Pairwise disjoint intersections among Steiner quadruple systems*, J.Inform.Optim.Sc. **9** (1988), 427–436.
9. G. Quattrocchi, *Intersection problems for STSs and SQSs: a short survey*, (to appear) (1989).