Indecomposable simple 2-(v, k, λ) designs of small orders

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Abstract. In this paper, simple 2-$(9, 4, \lambda)$ designs are constructed for $\lambda = 3q$, $1 \leq q \leq 7$, and indecomposable simple 2-(v, k, λ) designs areconstructed for it all $10 \leq v \leq 16$ and smallest possible λ for which the existence of simple 2-(v, k, λ) designs are previously undecided.

1. Introduction

A t-(v, k, λ) design is an ordered pair $(V, \mathbf{B})$ where V is a finite set containing v elements and $\mathbf{B}$ is a collection of k-subsets (called blocks) of V such that each t-subset of V is contained in exactly λ blocks. A 2-(v, k, λ) design is also known as a balanced incomplete block design.

A t-(v, k, λ) design is called simple if it contains no repeated blocks. A t-(v, k, λ) design $(V, \mathbf{B})$ is called decomposable or reducible if there is a sub-set $\mathbf{B}'$ of $\mathbf{B}$ such that $(V, \mathbf{B}')$ is a t-(v, k, λ') design for some $0 < \lambda' < \lambda$. A t-(v, k, λ) design is called indecomposable or irreducible if it is not decomposable.

The necessary conditions for the existence of a 2-(v, k, λ) design are

$$\begin{aligned} &\lambda(v-1) \equiv 0 \pmod{(k-1)}, \\ &\lambda v(v-1) \equiv 0 \pmod{k(k-1)}. \end{aligned} \tag{1}$$

It is well known ([5]) that for $k = 3, 4, 5$ and all λ, (1) is also sufficient for the existence of a 2-(v, k, λ) design with the exception $(v, k, \lambda) = (15, 5, 2)$, and for $k = 6$ and $\lambda \geq 2$, (1) is also sufficient for the existence of a 2-$(v, 6, \lambda)$ design wilh the exception $(v, k, \lambda) = (21, 6, 2)$.

It is not difficult to verify that the following is an additional condition for the existence of a simple 2-(v, k, λ) design:

$$\lambda \leq \binom{v-2}{k-2}. \tag{2}$$

In the case $k = 3$, it is proved ([6]) that (1) and (2) are also sufficient for the existence of a simple 2-$(v, 3, \lambda)$ design.

H.-D.O.F. Granau ([4]) listed the known results on the existence of simple t-(v, k, λ) designs for $6 \leq v \leq 16$. In the range $6 \leq v \leq 9$, the existence of simple 2-$(9, 4, 3q)$ for $2 \leq q \leq 5$ remains unknown. In the range $10 \leq v \leq 16$, there are 13 parameter triples (v, k, λ) for the smallest possible λ for which the existence of indecomposable simple 2-(v, k, λ) designs remains unknown.

It is the purpose of the present paper to prove that there exists a simple 2-$(9, 4, 3q)$ design for each q, $41 \leq q \leq 7$, and there exists an indecomposable simple 2-(v, k, λ) design for each of the 13 parameter triples (v, k, λ) for which the existence of a simple 2-(v, k, λ) design is previously undecided.

2. Construction of simple 2-$(9,4,3q)$ designs

Let $(Z_v, \mathbf{B})$ be a t-(v,k,λ) design. For $B = \{a_1, a_2, \cdots, a)k\} \in \mathbf{B}$, let

$$B + x = \{a_1 + x, a_2 + x, \cdots, a_k + x\}, \quad x \in Z_v$$

If for each $B \in \mathbf{B}$ and each $x \in Z_v$, we have $B + x \in \mathbf{B}$, then $(Z_v, \mathbf{B})$ is called a cyclic t-(v,k,λ) design.

The necessary condition for the existence of a simple 2-$(9,4,\lambda)$ design is $\lambda = 3q$, $1 \leq q \leq 7$. We will prove that this condition is also sufficient. In fact, we will prove the following stronger result:

Lemma 1. *There exists a cyclic simple* 2-$(9,4,3q)$ *design for each* q, $1 \leq q \leq 7$.

Proof: We construct three $(9,4,3)$ difference families and a $(9,4,6)$ difference family on Z_9 as follows:

$$D_1 = \{\{0,1,2,5\},\{0,1,4,6\}\};$$
$$D_2 = \{\{0,1,2,5\},\{0,1,3,7\}\};$$
$$D_3 = \{\{0,1,3,4\},\{0,1,3,5\}\};$$
$$D_4 = \{\{0,1,2,3\},\{0,1,4,5\},\{0,1,4,7\},\{0,2,4,6\}\}.$$

From these differences, we obtain three cyclic 2-$(9,4,3)$ designs, denoted $(Z_9, \mathbf{B}_1)$, $(Z_9, \mathbf{B}_2)$, $(Z_9, \mathbf{B}_3)$ respectively, and a cyclic 2-$(9,4,6)$ design, denoted $(Z_9, \mathbf{B}_4)$. It can be checked that all the orbits of the base blocks of D_1, D_2, D_3 and D_4 under the action of the additive group Z_9 are disjoint, so the 2-$(9,4,3)$ designs and the 2-$(9,4,6)$ design obtaincd above are simple. Let $\mathbf{B}_5 = \mathbf{B}_1 \cup \mathbf{B}_4$, $\mathbf{B}_6 = \mathbf{B}_2 \cup \mathbf{B}_5$, $\mathbf{B}_7 = \mathbf{B}_3 \cup \mathbf{B}_6$; then $(Z_9, \mathbf{B}_5)$ is a cyclic 2-$(9,4,9)$design and $(Z_9, \mathbf{B}_6)$ is a 2-$(9,4,12)$ design. Let $\mathbf{B}$ denote the set of all the 4-subsets of Z_9; then $(Z_9, \mathbf{B})$ is a cyclic simple 2-$(9,4,21)$ design, $(Z_9, \mathbf{B} \setminus \mathbf{B}_1)$ is a cyclic simple 2-$(9,4,18)$ design and $(Z_9, \mathbf{B}_7)$ is a cyclic simple 2-$(9,4,15)$.

3. Indecomposable simple 2-(v,k,λ) designs for $10 \leq v \leq 16$

In this section, we will construct simple 2-(v,k,λ) designs for $10 \leq v \leq 16$ and the smallest possible λ, whose existence was previously unknown. As λ is the smallest possible for the existence of a 2-(v,k,λ) design, the designs constructed in this section must be indecomposable.

Lemma 2. *There exists a cyclic indecomposable simple* 2-$(11,4,6)$ *design.*

Proof: It is easy to verify that the following is a $(11,4,6)$ difference family on

$$D: \{0,1,2,4\},\{0,1,2,6\},\{0,1,3,6\},\{0,1,5,8\},\{0,2,4,8\}$$

Z_{11} can be regarded as a group acting on the set of 4-subsets of Z_{11}. The orbits of the above 5 base blocks under lhe action of Z_{11} are disjoint, thus we ohtain a cyclic simple 2-$(11,4,6)$ design. As 6 is smallest possible for the existence of a 2-$(11,4,\lambda)$ design, this design is indecomposable.

Lemma 3. *There exists a cyclic indecomposable simple* 2-(13, 5, 5) *design.*

Proof: The following is a (13, 5, 5) difference family in Z_{13} and the base blocks under the action of Z_{13} are disjoint:

$$D : \{0,1,6,8,10\}, \{0,1,3,9,10\}, \{0,1,3,5,6\}$$

As there doesn't exist a 2-(13, 5, λ) design for $1 \leq \lambda < 5$, we obtain a cyclic indccomposable simple 2-(13, 5, 5) design.

Lemma 4. *There exists a cyclic indecomposable simple* 2-(15, 4, 6) *design and a cyclic indecomposable simple* 2-(15, 6, 5) *design.*

Proof: We construct a (15, 4, 6) difference family D_1 and a 2-(15, 6, 5) difference family D_2 on Z_{15} as follows:

$$D_1 : \{0,1,3,7\}, \{0,1,3,10\}, \{0,1,2,11\}, \{0,1,4,6\}, \{0,1,4,9\},$$
$$\{0,1,5,7\}, \{0,2,8,12\}$$
$$D_2 : \{0,1,3,4,10,12\}, \{0,1,2,3,7,11\}, \{0,2,5,7,10,12\}$$

The orbits of the base blocks in each difference family under the action of Z_{15} are disjoint, so we obtain a cyclic indecomposable simple 2-(15, 4, 6) design and a cyclic indecomposable simple 2-(15, 6, 5) design from D_1 and D_2 respectively.

It is convenient in some cases to construct 2-(v, k, λ) designs on $Z_{v-1} \cup \{\infty\}$ instead of Z_v, where ∞ is an element which is fixed under the action of the additive group Z_{v-1}. In this case we define

$$x + \infty = \infty, \quad \text{for } x \in Z_{v-1}$$

and consider difference families on $Z_{v-1} \cup \{\infty\}$. The designs obtained by developing the base blocks (mod $v - 1$) are called rotational.

Lemma 5. *There exists a rotational indecomposable simple* 2-(12, 5, 20) *design.*

Proof: The base blocks of the desired 2-(12, 5, 20) design on $Z_{11} \cup \{\infty\}$ are:

$$\{0,1,2,3,8\}, \{0,1,4,6,8\}, \{0,1,2,3,7\}, \{0,1,4,7,9\}$$
$$\{0,1,2,5,9\}, \{0,1,3,5,6\}, \{0,1,2,4,7\}, \{0,1,2,4,\infty\},$$
$$\{0,1,2,6,\infty\}, \{0,1,3,6,\infty\}, \{0,1,5,8,\infty\}, \{0,2,4,8,\infty\}.$$

Thc orbits of the base blocks under the action of Z_{11} are disjoint, so the 2-(12, 5, 20) design is simple, and as there is no 2-(12, 5, λ) design if $1 \leq \lambda \leq 19$, it is also indecomposable.

Lemma 6. *There exists an indecomposable simple* 2-$(14, k, \lambda)$ *design for each* $(k, \lambda) = (3,6), (4,5), (5,20)$ *or* $(6,15)$.

Proof: We construct a rotational indecomposable simple 2-$(14, k, \lambda)$ design on $Z_{13} \cup \{\infty\}$ for each of the parameter pairs $(k, \lambda) = (3,6), (4,6), (5,20)$ or $(6,15)$.

(i) $(k, \lambda) = (3,6)$. The base blocks of the indecomposable simple 2-$(14,3,6)$ design on $Z_{13} \cup \{\infty\}$ are:

$$\{0,1,2\}, \{0,3,5\}, \{0,4,8\}, \{0,1,3\}, \{0,1,4\}, \{0,1,5\}, \{0,1,6\},$$
$$\{0,2,5\}, \{0,2,6\}, \{0,2,7\}, \{0,3,7\}, \{0,2,\infty\}, \{0,5,\infty\}, \{0,6,\infty\}.$$

(ii) $(k, \lambda) = (4,6)$. The base blocks of the indecomposable simple 2-$(14,4,6)$ design on $Z_{13} \cup \{\infty\}$ are:

$$\{0,1,2,10\}, \{0,1,3,9\}, \{0,1,3,10\}, \{0,1,5,8\},$$
$$\{0,1,4,6\}, \{0,2,6,\infty\}, \{0,2,7,\infty\}.$$

(iii) $(k, \lambda) = (5,20)$. The base blocks of the indecomposable simple 2-$(14,5,20)$ design on $Z_{13} \cup \{\infty\}$ are:

$$\{0,1,2,3,5\}, \{0,2,4,6,10\}, \{0,4,7,11\}, \{0,1,3,6,11\},$$
$$\{0,1,2,8,9\}, \{0,1,2,6,7\}, \{0,1,3,8,11\}, \{0,1,4,8,9\},$$
$$\{0,1,3,7,10\}, \{0,1,2,6,\infty\}, \{0,1,4,8,\infty\}, \{0,1,4,9,\infty\},$$
$$\{0,2,4,6,\infty\}, \{0,1,3,11,\infty\}.$$

(iv) $(k, \lambda) = (6,15)$. The base blocks of the indecomposable simple 2-$(14,6,5)$ design on $Z_{13} \cup \{\infty\}$ are:

$$\{0,1,2,5,8,10\}, \{0,1,2,5,6,11\}, \{0,1,3,4,8,11\}, \{0,1,3,5,6,9\},$$
$$\{0,1,2,7,9,\infty\}, \{0,1,2,4,6,\infty\}, \{0,1,2,4,7,\infty\}.$$

Lemma 7. *There exists an indecomposable simple* 2-$(16,7,14)$ *design.*

Proof: We construct a cyclic indecomposable simple 2-$(16,7,14)$ design on Z_{16}; the base blocks are:

$$\{0,1,2,4,5,8,10\}, \{0,1,2,3,7,9,13\}, \{0,1,2,4,5,9,11\},$$
$$\{0,1,2,4,5,11,13\}, \{0,1,2,5,7,10,13\}.$$

It is proved ([7]) thal is $q \equiv 1 \pmod{k}$ is a prime power, then there exists a $(q, k, k-1)$-difference family on $GF(q)$, the finite field of order q. The construction is as follows:

Let x denote a primitive element of $GF(q)$. Let $e = (q-1)/k$, $\epsilon = x^e$ and $f = (q-1)/e$. Let $A = \{1, \epsilon, \epsilon^2, \cdots, \epsilon^{f-1}\}$, then

$$D = \{Aw^i / i = 0, 1, 2, \cdots, e-1\}$$

is a $(q, k, k-1)$-difference family of $GF(q)$.

In fact, it can be checked that the orbits of the base blocks $Aw^i (i = 0, 1, \cdots, e-1)$ under the action of the additive group of $GF(q)$ are disjoint. Thus the cyclic 2-$(q, k, k-1)$ design obtained from this difference family is simple, so we obtain the following result.

Lemma 8. *If* $q \equiv 1 (\bmod k)$ *is a prime power, then there exists a cyclic simple* 2-$(q, k, k-1)$ *design.*

Considering the necessary conditions of a 2-(v, k, λ) design, we obtain the following corollary.

Corollary. *There exists a cyclic indecomposable simple* 2-$(13, 6, 5)$ *design and a cyclic indecomposable simple* 2-$(16, 5, 4)$ *design.*

We note at the end of this section that an indecomposable simple 2-$(12, 4, 3)$ design can be found in ([1]).

4. Summary

Combining the results obtained in this paper and the results previously known (see [4]), we have already completely detemined the existence of indecomposable simple 2-(v, k, λ) designs for $10 \leq v \leq 16$ and the smallest possible λ. We collect these results in the following table. For the previously known results, the references can be found in ([4], Table 2).

Table. Indecomposable simple 2-(v, k, λ) designs with $10 \leq v \leq 16$.

No.	(v, k, λ)	Existence	Reference
1	(10,3,2)	yes	
2	(10,4,2)	yes	
3	(10,5,4)	yes	
4	(11,3,3)	yes	
5	(11,4,6)	yes	Lemma 2
6	(11,5,2)	yes	
7	(12,3,2)	yes	
8	(12,4,3)	yes	[1]
9	(12,5,20)	yes	Lemma 5
10	(12,6,5)	yes	
11	(13,3,1)	yes	
12	(13,4,1)	yes	
13	(13,5,5)	yes	Lemma 3
14	(13,6,5)	yes	Lemma 8
15	(14,3,6)	yes	Lemma 6
16	(14,4,6)	yes	Lemma 6
17	(14,5,20)	yes	Lemma 6
18	(14,6,15)	yes	Lemma 6
19	(14,7,6)	yes	
20	(15,3,1)	yes	
21	(15,4,6)	yes	Lemma 4
22	(15,5,2)	no	
23	(15,6,5)	yes	Lemma 4
24	(15,7,3)	yes	
25	(16,3,2)	yes	
26	(16,4,1)	yes	
27	(16,5,4)	yes	Lemma 8
28	(16,6,1)	no	
29	(16,7,14)	yes	Lemma 7
30	(16,8,7)	yes	

It is worth remarking that all the results in this paper were obtained by hand calculations.

References

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