ON THE PRODUCT OF k-MINIMAL DOMINATION NUMBERS OF A GRAPH AND ITS COMPLEMENT

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Abstract. The *domination number* $\gamma(G)$ of a graph $G = (V, E)$ is the smallest cardinality of a *dominating* set X of G, i.e. of a subset X of vertices such that each $v \in V - X$ is adjacent to at least one vertex of X.

The *k-minimal domination number* $\Gamma_k(G)$, is the largest cardinality of a dominating set Y which has the following additional property: For every ℓ-subset Z of Y where $\ell \leq k$ and each $(\ell - 1)$-subset W of $V - Y$, the set $(Y - Z) \cup W$ is not dominating.

In this paper, for any positive integer $k \geq 2$, we exhibit a self-complementary graph G with $\gamma(G) > k$ and use this and a method of Graham and Spencer to construct n-vertex graphs F for which $\Gamma_k(F)\Gamma_k(\overline{F}) > n$.

1. INTRODUCTION

A subset X of vertices of a graph $G = (V, E)$ is a *dominating set* if each $v \in V - X$ is adjacent to at least one vertex of X. The *domination number* $\gamma(G)$ (*upper domination number* $\Gamma(G)$) of G is the smallest (largest) cardinality of a minimal dominating set of G. The reader is referred to [6] for an excellent bibliography concerning the theory of dominating sets of graphs. Bollobás, Cockayne and Mynhardt [1] generalised these concepts by extending the idea of minimality. The *k-minimal domination number* $\Gamma(G)$ is the largest cardinality of a *k-minimal dominating set* of G, i.e., a dominating set Y of G with the following additional property: For every subset Z of Y, where $\ell \leq k$, and each $(\ell - 1)$ subset W of $V - Y$, the set $(Y - Z) \cup W$ is not a dominating set of G. We note that 1-minimality is precisely minimality and that

$$\gamma(G) \leq \cdots \leq \Gamma_k(G) \leq \Gamma_{k-1}(G) \leq \cdots \leq \Gamma_2(G) \leq \Gamma_1(G) = \Gamma(G). \quad (1)$$

The values of $\Gamma_k(G)$ where G is a path or a cycle are calculated in [1, 3].

Bounds of the form $\mu(G)\mu(\overline{G}) \leq f(n)$ concerning the parameter of the n vertex graph G and its complement $\overline{G}$, have been called Nordhaus-Gaddum results due to their theorem of this type concerning the chromatic number (see [8]). Several authors have found Nordhaus-Gaddum results concerning dominating sets. Jaegar and Payan [7] proved that for any n-vertex graph G, $\gamma(G)\ \gamma(\overline{G}) \leq n$ and the extremal graphs for this inequality were found by Payan and Xuong [9]. In [1], this inequality was improved to $\Gamma_2(G)\gamma(\overline{G}) \leq n$. Cockayne and Mynhardt [4] showed that $\Gamma(G)\Gamma(\overline{G}) \leq \left\lceil \frac{n^2+2n}{4} \right\rceil$ and established the extremal graphs.

In view of the inequalities (1) it is natural to ask for the maximum value $r(n, k)$ of $\Gamma_k(G)\Gamma_k(\overline{G})$ for an n-vertex graph G. The above discussion immediately gives

$$n \leq r(n, k) \leq \left\lceil \frac{n^2 + 2n}{4} \right\rceil$$

The determination of $r(n, k)$ appears to be a formidable task and remains an open question. In this paper, by explicit construction, we show that for any $k \geq 2$, $r(n, k) > n$. The process involves the construction of self-complementary graphs whose domination number exceeds any given integer k. These graphs are exhibited in the next section.

2. THE GRAPH $G_{p,k}$

Let k be an integer greater than 1. In this section we exhibit a p-vertex self-complementary graph $G_{p,k}$ whose domination number exceeds k. We emulate the techniques used by Graham and Spencer [5] in their work concerning dominating sets of tournaments.

Let p be a prime such that $p \equiv 1 \pmod 4$ and $V(G_{p,k}) = 0, \ldots, p-1 = Z_p$. Two vertices a, b are adjacent if and only if $a - b$ is not a quadratic residue of p. The graph $\overline{G_{p,k}}$ (i.e. two vertices a, b of $\overline{G_{p,k}}$ are adjacent if and only if $a - b$ is a quadratic residue of p) is called a *Paley graph* (see e.g. [2, p. 345]). It is easily seen, using the transformation $x \to \lambda x$ where λ is any quadratic non-residue, that $G_{p,k}$ is self-complementary and hence also a Paley graph. We now show that for p sufficiently large, $\gamma(G_{p,k}) > k$.

Theorem 1. *For $k \geq 2$ and $p > k^2 2^{2k-2}$, $\gamma(G_{p,k}) > k$.*

Proof: Following [5], for $a \in Z_p$ we define $\chi(a) = 1(-1)$ if a is (is not) a quadratic residue of p and $\chi(0) = 0$. A set $A = \{a_1, \ldots, a_k\}$ does not dominate the vertex x if for all $j = 1, \ldots, k$, $\chi(x - a_j) = 1$. Define $g(A)$ by

$$g(A) = \sum_{\substack{x=0 \\ x \notin A}}^{p-1} \prod_{j=1}^{k} = [1 + \chi(x - a_j)] .$$

It is sufficient to show that $g(A) > 0$ for any A; for in this case there exists $x_0 \notin A$ such that

$$\prod_{j=1}^{k} [1 + \chi(x_0 - a_j)] > 0$$

and hence $\chi(x_0 - a_j) = 1$ for each $j = 1, \ldots, k$, i.e. A does not dominate x_0. Let

$$h(A) = \sum_{x=0}^{p-1} \prod_{j=1}^{k} [1 + \chi(x - a_j)] .$$

Then, exactly as in [5], we have

$$g(A) = h(A) - \sum_{i=1}^{k} \prod_{j=1}^{k} [1 + \chi(a_i - a_j)] \tag{2}$$

and

$$h(A) = \sum_{x=0}^{p-1} 1 + \sum_{x=0}^{p-1} \sum_{j=1}^{k} \chi(x - a_j)$$
$$\ldots + \sum_{x=0}^{p-1} \sum_{j_1<j_2} \chi(x - a_{j_1})\chi(x - a_{j_2}) + \ldots$$
$$\ldots + \sum_{x=0}^{p-1} \sum_{j_1<\ldots<j_s} \chi(x - a_{j_1}) \ldots \chi(x - a_{j_s}) + \ldots$$
$$\ldots + \sum_{x=0}^{p-1} \sum_{j_1<\ldots<j_k} \chi(x - a_{j_1}) \ldots \chi(x - a_{j_k}). \tag{3}$$

The first term of (3) is p and since $G_{p,k}$ is regular of degree $(p-1)/2$, the second term is 0. Thus the analysis may proceed exactly as the relations (7)–(11) of [5] and we conclude

$$h(A) \geq p - \left[(k-2)2^{k-1} + 1\right]\sqrt{p}. \tag{4}$$

If i is fixed, $1 + \chi(a_i - a_j) = 1$ for $i = j$ and is at most 2 otherwise. Hence

$$\prod_{j=1}^{k} [1 + \chi(a_1 - a_j)]$$

is at most 2^{k-1}. It follows from (2) that $h(A) - g(A) \leq k2^{k-1}$ so that from (4)

$$g(A) \geq p - \left[(k-2)2^{k-1} + 1\right]\sqrt{p} - k \cdot 2^{k-1}. \tag{5}$$

It is easy to show that the right hand side of (5) is positive for $p \geq (k2^{k-1})^2$ and $k \geq 2$, hence $\gamma(G_{p,k}) > k$ for these values, as required.

3. GRAPHS WITH $\Gamma_k(G)\Gamma_k(\overline{G}) > n$

Let G and H be graphs of order m and q respectively, with $V(G) = \{v_1, \ldots, v_m\}$. Then $G \oplus H$ denotes the graph obtained by replacing each vertex v_i of G with a copy H_i of H and each edge v_iv_j of G with $K_{q,q}$, where the edges of $K_{q,q}$ join the vertices of H_i to the vertices of H_j.

Let $V(H_i) = \{v_{i1}, \ldots, v_{iq}\}$, $i = 1, \ldots, m$ and $G * H$ be the graph obtained from $G \oplus H$ in the following way: If $v_iv_j \in E(G)$ ($v_iv_j \notin E(G)$ respectively), remove (add) the set $E_{ij} = \{v_{i\ell}v_{j\ell}/\ell = 1, \ldots, q\}$ of edges from (to) $G \oplus H$, $i, j = 1, \ldots, m$. We observe that if G and H are self-complementary graphs, then $G * H$ is also self-complementary.

Lemma 1. *If $\gamma(G) \geq 2$ and $q \geq 2$, then each $V(H_i)$, $i = 1, \ldots, m$, is a minimal dominating set of $G * H$.*

Proof: Without loss of generality we prove that $V(H_1)$ is a minimal dominating set of $G * H$. Let

$$V_1 = \bigcup_{\substack{i=2 \\ v_1 v_i \in E(G)}}^{m} V(H_i)$$

and

$$V_2 = \bigcup_{\substack{i=2 \\ v_1 v_i \notin E(G)}}^{m} V(H_i).$$

Any vertex $v_{1\ell}$ in $V(H_1)$ dominates all vertices in V_1 except those labelled $v_{i\ell}$ for some i; hence any two vertices in $V(H_1)$ dominate V_1. On the other hand, each vertex $v_{1\ell} \in V(H_1)$ dominates exactly those vertices of V_2 that are labelled $v_{i\ell}$ for some i; hence $V(H_1)$ dominates V_2 and no subset of $V(H_1)$ dominates V_2. Since $\gamma(G) \geq 2$, $V_2 \neq \emptyset$ and hence $V(H_1)$ is a minimal dominating set of $G * H$. ∎

Lemma 2. *If D is a dominating set of $G * H$, then at least one of the following holds:*

(i) $|D| \geq q$;
(ii) $V(H_i) \cap D \neq \emptyset$ *for at least* $\gamma(G)$ *copies* H_i *of* H *in* $G * H$.

Proof: Suppose D is a dominating set of $G*H$ such that (ii) above is not satisfied. Without loss of generality, assume that $H_1, \ldots, H_r$, $r < \gamma(G)$, are the copies of H in $G * H$ for which $V(H_i) \cap D \neq \emptyset$ and let $T = \{v_1, \ldots, v_r\}$ be the vertices of G corresponding to $H_1, \ldots, H_r$. Since $r < \gamma(G)$, there is a vertex $v_s \in V(G)$, $s > r$, which is not dominated in G by T. Hence, by the construction of $G * H$, each vertex in $\cup_{i=1}^{r} V(H_i)$ is adjacent to exactly one vertex of H_s and since $D \subseteq \cup_{i=1}^{r} V(H_i)$ and D dominates H_s, this implies that $|D| \geq q$. ∎

We now state and prove the principal result.

Theorem 2. *Let $k \geq 2$ and let p be any prime such that $p \equiv 1 \pmod 4$ and $p > (k2^{k-1})^2$. Then, for any self-complementary graph H of order $q > p$, the n-vertex graph $G = G_{p,k} * H$ satisfies $\Gamma_k(G)\Gamma_k(\overline{G}) > n$.*

Proof: By Lemma 1, $V(H_i)$ is a minimal dominating set of G for each $i = 1, \ldots, p$. We prove that $V(H_1)$ (say) is also a k-minimal dominating set. Let $S \subseteq V(H_1)$ with $|S| = \ell \leq k$ and consider any $R \subseteq V(G) - V(H_1)$ with $|R| = \ell - 1$.

Then $X = (V(H_1) - S) \cup R$ has $|X| = q - 1$ and does not satisfy condition (i) of Lemma 2. Further, at most k copies of H in G contain vertices of X. Since

$\gamma(G_{p,k}) > k$, X does not satisfy condition (ii) of Lemma 2. We conclude that X does not dominate G. Hence $V(H_1)$ is k-minimal as asserted and $\Gamma_k(G) \geq q$. But $n = pq$, hence

$$\Gamma_k(G) \geq \sqrt{q}\sqrt{q} > \sqrt{p}\sqrt{q} = \sqrt{n}.$$

Since G is self-complementary, $\Gamma_k(G)\Gamma_k(\overline{G}) > n$. This completes the proof. ∎

We notice that by Lemma 2 and the construction of $G_{p,k} * H$, $\gamma(G_{p,k} * H) > k$. However, if $\gamma(G) < k$ for $k \geq 2$ and some n-vertex graph G, then no dominating set of G with more than $\gamma(G)$ vertices can be a k-minimal dominating set and hence $\Gamma_k(G) = \gamma(G)$. By using the fact that $\gamma(G)\Gamma_2(\overline{G}) \leq n$ (see [1]), it follows that in this case $\Gamma_k(G)\Gamma_k(\overline{G}) \leq n$.

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