Dominance Method for Plane Partitions. V: Enumeration of Ladder Tableaux

W. C. Chu

Institute of Systems Science
Academia Sinica, Peking 100080
People's Republic of CHINA

Abstract. The concept of ladder tableaux is introduced which may be considered as a natural extension of the shifted tableaux. By means of the dominance technique, a pair of determinantal expressions in terms of symmetric functions, for the generating function of ladder tableaux with a fixed shape, is established. As applications, the particular cases yield the generating functions for column-strict reverse plane partitions, symmetrical reverse plane partitions and column-strict shifted reverse plane partitions with a given shape and with no part-restrictions.

1. Preliminaries

Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \lambda_r)$ and $\nu = (\nu_1 \geq \nu_2 \geq \cdots \geq \nu_r)$ be two ordinary partitions satisfying $\nu_1 \leq \lambda_r$. Denote their conjugations by $\lambda' = (\lambda'_1 \geq \lambda'_2 \geq \cdots \geq \lambda'_c)$ and $\nu' = (\nu'_1 \geq \nu'_2 \geq \cdots \geq \nu'_c)$ respectively. The converse of ν is defined to be $\widetilde{\nu} = (\widetilde{\nu}_1 \geq \widetilde{\nu}_2 \geq \cdots \geq \widetilde{\nu}_r)$ whose conjugation is just ν' where $\widetilde{\nu}_i = \nu_{r-i+1}$ for $1 \leq i \leq r$. A ladder tableau of shape $\lambda/\widetilde{\nu}$ is an array T of positive integers $\{T_{ij}; \widetilde{\nu}_i < j < \lambda_i, 1 \leq i \leq r\}$ such that the entries of T are arranged in ascending order in each row and in strictly ascending order in each column, i.e., $t_{ij} \leq t_{i,j+1}$ for all $\widetilde{\nu}_i < j < \lambda_i$, $1 \leq i \leq r$ and $t_{ij} < t_{i+1,j}$ for all $1 \leq i < \lambda'_j - \nu'_j, \widetilde{\nu}_1 < j \leq c$. In this definition, the word "ladder" comes from the fact that if we turn over tableau T, then the shape of T is just like a ladder. For the tableau T with n_k parts equal k, its weight is defined to be a monomial $w(T) = \prod x_k^{n_k}$ which clearly makes sense for the tableaux with only one row (column). The enumerative function for any set P of tableaux is denoted by the weight-sum $\sum_{T \in P} w(T)$, which is polynomial or formal power series in variables $\{x_k\}_{k \geq 1}$.

In order to demonstrate the dominance technique, we need to introduce the following functions [15]: Let $e_k(u, v)$ denote the elementary symmetric function of degree k in variables $\{x_i\}_{u<i<v}$ if $v - u > k > 0$. If $k = 0$, then $e_k(u, v) = 1$ for $u < v$ and $e_k(u, v) = 0$ for $u \geq v$. If $k < 0$ or $v - u \leq k$ then $e_k(u, v) = 0$. Similarly, denote by $h_k(u, v)$ the complete homogenous symmetric function of degree k in variables $\{x_j\}_{u \leq j \leq v}$ if $u \leq v$ and $k > 0$. If $k = 0$ then $h_k(u, v) = 1$ for $u \leq v$ and $h_k(u, v) = 0$ for $u > v$. If $u > v$ or $k < 0$, then $h_k(u, v) = 0$. Analogous to those for ordinary and Gaussian binomial coefficients we have the following summation formulas.

Lemma 1.1.

$$\text{i.} \quad \sum_{m \le v < n} x_v e_k(u,v) = e_{k+1}(u,n) - e_{k+1}(u,m) \tag{1.1}$$

$$\text{ii.} \quad \sum_{m \le u < n} x_u h_k(u,v) = h_{k+1}(m,v) - h_{k+1}(n,v) \tag{1.2}$$

Proof: The generating functions for $\{e_k(u,v)\}$ and $\{h_k(u,u)\}$ with respect to the subscript k are respectively given by

$$E(u,v) = \sum_{k \ge 0} e_k(u,v) z^k = \prod_{u<i<v} (1 + z x_i)$$

and

$$H(u,v) = \sum_{k \ge 0} h_k(u,v) z^k = \prod_{u \le j \le v} (1 - z x_j)^{-1}.$$

Then (1.1) and (1.2) follow from the expansions of

$$\begin{aligned} \sum_{m \le v < n} x_v E(u,v) &= \sum_{m \le v < n} z^{-1}\{E(u,v+1) - E(u,v)\} \\ &= z^{-1}\{E(u,n) - E(u,m)\} \end{aligned}$$

and

$$\begin{aligned} \sum_{m \le u < n} x_u H(u,v) &= \sum_{m \le u < n} z^{-1}\{H(u,v) - H(u+1,v)\} \\ &= z^{-1}\{H(m,v) - H(n,v)\} \end{aligned}$$

respectively. ∎

Based on identities (1.1–1.2), the next two sections will establish the generating functions of flagged ladder tableaux as determinants in terms of elementary symmetric functions $e_k(u,v)$ and complete symmetric functions $h_k(u,v)$. Naturally, these determinantal expressions have resemblance as those for ordinary and skew tableaux with flags.

2. Row-Flagged Ladder Tableaux

Now we consider the row-flagged tableaux [15]. Denote by $S_{\lambda/\widetilde{\nu}}(t;b)$ the generating function for the ladder tableaux of shape $\lambda/\widetilde{\nu}$ with row flags t and b (i.e., the entries of ith row are bounded below by t_i and above by b_i) in which t_i becomes the first entry on the ith row of the augmented tableaux if attaching the flag t to the left of the original ones. It is obvious that the ladder tableaux enumerated by $S_{\lambda/\widetilde{\nu}}(t;b)$ consist of the entries $\{T\}$ which satisfy the following condition:

$$
\begin{array}{ccccccccccccccccccccccc}
t_1 & \leq & t_{1,\widetilde{\nu}_1+1} & \leq\cdots\leq & t_{1,\widetilde{\nu}_2} & \leq & t_{1,\widetilde{\nu}_2+1} & \leq\cdots\leq & t_{1,\widetilde{\nu}_r} & \leq & t_{1,\widetilde{\nu}_r+1} & \leq\cdots\leq & t_{1,\lambda_r} & \leq & t_{1,\lambda_r+1} & \leq\cdots\leq & t_{1,\lambda_2} & \leq & t_{1,\lambda_2+1} & \leq\cdots\leq & t_{1,\lambda_1} & \leq & b_1 \\
 & & & & \wedge & & \wedge & & \wedge & & \wedge & & \wedge & & \wedge & & \wedge & & & & & & \\
 & & & & t_2 & \leq & t_{2,\widetilde{\nu}_2+1} & \leq\cdots\leq & t_{2,\widetilde{\nu}_r} & \leq & t_{2,\widetilde{\nu}_r+1} & \leq\cdots\leq & t_{2,\lambda_r} & \leq & t_{2,\lambda_r+1} & \leq\cdots\leq & t_{2,\lambda_2} & \leq & b_2 & & & & \\
 & & & & & & & & \wedge & & \wedge & & \wedge & & \wedge & & & & & & & & \\
 & & & & & & & & \vdots & & \vdots & \cdots & \vdots & & \vdots & & & & & & & & \\
 & & & & & & & & \wedge & & \wedge & & \wedge & & & & & & (T) & & & & \\
 & & & & & & & & t_r & \leq & t_{r,\widetilde{\nu}_r+1} & \leq\cdots\leq & t_{r,\lambda_r} & \leq & b_r & & & & & & & &
\end{array}
\tag{2.1}
$$

For each tableau T with the restriction of (2.1), it can be regarded as a formal product of its successive columns $T_{\widetilde{\nu}_1+1} \leq T_{\widetilde{\nu}_1+2} \leq \cdots \leq T_{\lambda_1}$ after having defined an order relation "$\leq$" on column-tableaux: $T \leq T'$ if and only if the array formed by two columns is a ladder tableau. Thus $S_{\lambda/\widetilde{\nu}}(t;b)$ can be expressed as

$$
S_{\lambda/\widetilde{\nu}}(t;b) = \sum_{T_{\widetilde{\nu}_1+1} \leq T_{\widetilde{\nu}_1+2} \leq \cdots \leq T_{\lambda_1}} w(T_{\widetilde{\nu}_1+1})\, w(T_{\widetilde{\nu}_1+2})\, w(T_{\lambda_1}) \tag{2.2}
$$

where the summation runs over all T restricted by (2.1). This separation suggests the following recurrence.

Lemma 2.1. *$S_{\lambda/\widetilde{\nu}}(t;b)$ can be generated from the reverse recursive relations (where $c := \lambda_1$ and $e := \widetilde{\nu}_1$).*

$$
\left.
\begin{array}{l}
\textit{Initial condition} \\
\qquad U_{c+1}(T_c) = U_{c+1} = 1 \\
\textit{Recurrence relation} \\
\qquad U_k(T_{k-1}) = \displaystyle\sum_{T_k \text{ subject to (2.1)}} w(T_k)\, U_{k+1}(T_k) \\
\qquad\qquad (k = c, c-1, \cdots, e+1) \\
\textit{Final step} \\
\qquad S_{\lambda/\widetilde{\nu}}(t;b) = U_{e+1}.
\end{array}
\right\}
\tag{2.3}
$$

To carry on recursion (2.3), it may be convenient for one to add a natural restriction on row-flags as follows:

$$
\begin{array}{ccccccc}
0 < & t_1 & \leq & t_2 & \leq\cdots\leq & t_r \\
 & \wedge| & & \wedge| & & \wedge| \\
 & b_1 & \leq & b_2 & \leq\cdots\leq & b_r
\end{array}
\tag{2.4}
$$

Notice that the matrix $[h_{i-j+\lambda_j-\lambda_i}(t_{i,\lambda_i}, b_j)]_{1\le i,j\le r}$ being strictly lower unitriangular coincides with the fact that b is an upper row flag of T. Since recurrence (2.3) is restricted by table (2.1). Multiplying U_{c+1} by determinant

$$\det_{r\times r}[h_{i-j+\lambda_j-\lambda_i}(t_{i,\lambda_i}, b_j)] = 1$$

will gives that

$$U_c(T_{c-1}) = \sum_{\substack{t_{1c} < t_{2c} < \cdots < t_{\lambda'_c,c} \le b_{\lambda'_c} \\ \vee| \quad \vee| \quad\quad \vee| \\ t_{1,c-1} \quad t_{2,c-1} \quad \cdots \quad t_{\lambda'_c,c-1}}} \det_{r\times r}\begin{bmatrix} x_{t_{i,c}} h_{i-j+\lambda_j-\lambda_i}(t_{i,c}, b_j) \\ h_{i-j+\lambda_j-\lambda_i}(t_{i,\lambda_i}, b_j)\end{bmatrix} \begin{matrix} 1\le i\le \lambda'_c \\ \lambda'_c < i \le r\end{matrix} \tag{2.5}$$

If performing the summation with respect to $t_{i,c}$, the ith row of the determinant in (2.5) becomes, by (1.2),

$$h_{i-j+\lambda_j-\lambda_i+1}(t_{i,c-1}, b_j) - h_{i-j+\lambda_j-\lambda_i+1}(t_{i+1,c}, b_j), \quad j = 1, 2, \cdots, r.$$

Thus multiplying the $(i+1)$th row by $x^{-1}_{t_{i+1,c}}$ and adding to the ith row, the latter reduces to

$$h_{i-j+\lambda_j-\lambda_i+1}(t_{i,c-1}, b_j), \quad j = 1, 2, \cdots, r.$$

Performing the operations in this way with subscript i from 1 to λ'_c we finally arrive at

$$U_c(T_{c-1}) = \det_{r\times r}\begin{bmatrix} h_{i-j+\lambda_j-\lambda_i+1}(t_{i,c-1}, b_j) \\ h_{i-j+\lambda_j-\lambda_i}(t_{i,\lambda_i}, b_j)\end{bmatrix} \begin{matrix} 1\le i\le \lambda'_c \\ \lambda'_c < i \le r\end{matrix} \tag{2.6}$$

For the recurrence (2.3) if $x_{t_{i,c-1}}$ being absorbed in the ith row of determinant (2.6), then the similar manner yields

$$U_c(T_{c-1}) = \det_{r\times r}\begin{bmatrix} h_{i-j+\lambda_j-\lambda_i+2}(t_{i,c-2}, b_j) \\ h_{i-j+\lambda_j-\lambda_i+1}(t_{i,c-2}, b_j) \\ h_{i-j+\lambda_j=\lambda_i}(t_{i,\lambda_i}, b_j)\end{bmatrix} \begin{matrix} 1\le i\le \lambda'_c \\ \lambda'_c < i \le r \\ \lambda'_{c-1} < i \le r\end{matrix}$$

Repeating this process $(c-e)$ times on recurrence (2.3), the intermediate results are demonstrated as follows:

$$U_{\lambda k+1}(T_{\lambda k}) = \det_{r\times r}\begin{bmatrix} h_{i-j+\lambda_j-\lambda_k}(t_{i,\lambda_k}, b_j) \\ h_{i-j+\lambda_j-\lambda_i}(t_{i,\lambda_i}, b_j)\end{bmatrix} \begin{matrix} 1\le i\le k \\ k < i \le r\end{matrix} \quad (k = 1, 2, \cdots, r)$$

$$U_{\widetilde{\nu}_k+1}(T_{\widetilde{\nu}_k}) = \det_{r\times r}\begin{bmatrix} h_{i-j+\lambda_j-\widetilde{\nu}_k}(t_{i,\widetilde{\nu}_k}, b_j \\ h_{i-j+\lambda_j-\widetilde{\nu}_i}(t_i, b_j)\end{bmatrix} \begin{matrix} 1\le i< k \\ k \le i < r\end{matrix} \quad (k = r, r-1, \cdots, 1)$$

Thus we ultimately obtain the following

Theorem 2.2. *The enumerative function for the ladder tableaux defined by table (2.1) is given by flagged-Schur function*

$$S_{\lambda/\tilde{\nu}}(t;b) = \det_{r\times r}\left[h_{i-j+\lambda_j-\tilde{\nu}_i}(t_i,b_i)\right]. \tag{2.7}$$

Remarkably, if replacing inverse partition $\tilde{\nu}$ by a ordinary partition μ, then $S_{\lambda/\mu}(a;b)$ coincides with the generating function of skew tableaux of shape λ/μ with row flags a and b (cf. sec. 5.4.5 in [5] and [15]).

3. Column-Flagged Ladder Tableaux

Without loss of generality we will assume that $\tilde{\nu}_1 = 1$ in this section. Define the sequence $tVb = (t_1, t_2, \cdots, t_{\tilde{\nu}_r}; b_{\tilde{\nu}_r+1}, b_{\tilde{\nu}_r+2}, \cdots, b_c)$ by $(tVb)_k$ equal t_k for $1 \le k \le \tilde{\nu}_r$ and b_k for $\tilde{\nu}_r < k \le c$ respectively, where $c = \lambda_1$ as in lemma 2.1. Denote by $\underline{S}_{\lambda/\tilde{\nu}}(a;tVb)$ the generating function for the lader tableaux of shape $\lambda/\tilde{\nu}$ with column-flags a and tVb (i.e., the entries of column j are strictly bounded below by a_j and above by $(tVb)_j$) in which t_j becomes the last entry on the jth column of the augmented tableaux if we regard t as complements to the original tableaux. Hence the ladder tableaux enumerated by $\underline{S}_{\lambda/\tilde{\nu}}(a;tVb)$ consist of the arrays $\{T\}$ subject to the conditions (where it is convenient for one to attach a redundant column $a_1 < t_1$)):

$$\begin{array}{ccccccccccccccccccc}
a_1 & a_2 & \cdots & a_{\tilde{\nu}_2} & a_{\tilde{\nu}_2+1} & \cdots & a_{\tilde{\nu}_r} & a_{\tilde{\nu}_r+1} & \cdots & a_{\lambda_r} & a_{\lambda_r+1} & \cdots & a_{\lambda_2} & a_{\lambda_2+1} & \cdots & a_c \\
\wedge & \wedge & & \wedge & \wedge & & \wedge & \wedge & & \wedge & \wedge & & \wedge & \wedge & & \wedge \\
t_1 \le & t_{1,2} \le & \cdots \le & t_{1,\tilde{\nu}_2} \le & t_{1,\tilde{\nu}_2+1} \le & \cdots \le & t_{1,\tilde{\nu}_r} \le & t_{1,\tilde{\nu}_r+1} \le & \cdots \le & t_{1,\lambda_r} \le & t_{1,\lambda_r+1} \le & \cdots \le & t_{1,\lambda_2} \le & t_{1,\lambda_2+1} \le & \cdots \le & t_{1,c} \\
 & \wedge & & \wedge & \wedge & & \wedge & \wedge & & \wedge & \wedge & & \wedge & \wedge & & \wedge \\
 & t_2 \le & \cdots \le & t_{\tilde{\nu}_2} \le & t_{2,\tilde{\nu}_2+1} \le & \cdots \le & t_{2,\tilde{\nu}_r} \le & t_{2,\tilde{\nu}_r+1} \le & \cdots \le & t_{2,\lambda_r} \le & t_{2,\lambda_r+1} \le & \cdots \le & t_{2,\lambda_2} & b_{\lambda_2+1} & \cdots & b_c \\
 & & & & \wedge & & \wedge & \wedge & & \wedge & \wedge & & \wedge & & & \\
 & & & & t_{\tilde{\nu}_2+1} & & \vdots & \vdots & \cdots & \vdots & \vdots & & b_{\lambda_2} & & & \\
 & & & & & \ddots & \wedge & \wedge & & \wedge & \wedge & & & & & \\
 & & & & & & t_{\tilde{\nu}_r} \le & t_{r,\tilde{\nu}_r+1} \le & \cdots \le & t_{r,\lambda_r} & b_{\lambda_r+1} & & & & & \\
 & & & & & & & \wedge & & \wedge & & & & & & \\
 & & & & & & & b_{\tilde{\nu}_r+1} & \cdots & b_{\tilde{\nu}_r} & & & & (T) & &
\end{array} \tag{3.1}$$

Notice that each tableau T defined by (3.1) can be considered as a formal product of its succesive rows $T_1 < T_2 < \cdots < T_r$ after having defined an order relation "$<$" on row-tableaux: $T < T'$ if and only if the array formed by two rows is a ladder tableau. Thus $\underline{S}_{\lambda/\tilde{\nu}}(a;tVb)$ can be represented as

$$\underline{S}_{\lambda/\tilde{\nu}}(a;tVb) = \sum_{T_1<T_2<\cdots<T_r} w(T_1)w(T_2)\cdots w(T_r) \tag{3.2}$$

where the summation runs over all T subject to (3.1). This expression suggests the following

Lemma3.1. *$\underline{S}_{\lambda/\widetilde{\nu}}(a;tVb)$ can be derived recurssively from the iterative process:*

$$\left.\begin{array}{l} \textit{Initial condition (where } t_{1,1}=t_1\textit{)} \\ \quad V_0(T_1) = \det_{c\times c}\,[\,e_{j-i}(a_j,t_{1,i})\,] = 1 \\ \textit{Recurrence relation} \\ \quad V_k(T_{k+1}) = \sum_{T_k \text{ subject to } (3.1)} w(T_k)\,V_{k-1}(T_k) \\ \qquad (k=1,2,\cdots,r) \\ \textit{Final step} \\ \quad \underline{S}_{\lambda/\widetilde{\nu}}(a;tVb) = V_r \end{array}\right\} \tag{3.3}$$

By means of (1.1), manipulate on the determinant with respect to each row T_k from right to left in reccurrence (3.3). The exactly same procedure as that for (2.7) demonstrated in the last section will produce the dual proposition of theorem 2.2 as long as column-flags satisfy the following conditions

$$\begin{array}{l} a_1 \le a_2 \le \cdots \le a_{\widetilde{\nu}_r} \le a_{\widetilde{\nu}_r+1} \le \cdots \le a_c \\ \wedge \quad \wedge \qquad\quad \wedge \quad\ \wedge \qquad\qquad \wedge \\ t_1 \le t_2 \le \cdots \le t_{\widetilde{\nu}_r} \le b_{\widetilde{\nu}_r+1} \le \cdots \le b_c \end{array} \tag{3.4}$$

In fact the restriction on tVb can be replaced by more natural ones

$$\begin{array}{ll} t_k + \widetilde{\nu}'_k \le t_{k+1} + \widetilde{\nu}'_{k+1} & (1 \le k \le \widetilde{\nu}_r) \\ b_k - \lambda'_k \le b_{k+1} - \lambda'_{k+1} & (\widetilde{\nu}_r < k \le c) \end{array} \tag{3.5}$$

which come from the considerations for the extreme cases of corner-hooks.

Theorem 3.2. *With column-flags defined by (3.4) (resp. tVb by (3.5)), the generating function for the ladder tableaux of shape $\lambda/\widetilde{\nu}$ with column-flags a and tVb is given by*

$$\underline{S}_{\lambda/\widetilde{\nu}}(a;tVb) = \det_{c\times c}\,[\,e_{j-i+\lambda'_i-\nu'_i}(a_j,(tVb)_i)\,]\,. \tag{3.6}$$

When $\nu = 0$, this determinant reduces to the flagged Schur function [15].

4. Applications to Column-Strict Reverse Plane Partitions

To discuss the enumeration problems of reverse plane partitions, we first recall some identities about Schur functions which may be refered to [8], [9], and [12]. For $\delta = (r-1, r-2, \cdots, 1, 0)$ and partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_r)$ defined in Section 1, Schur function has a quotient expression in determinants:

$$S_\lambda(x_1,x_2,\cdots,x_r) = \det_{r\times r}\,[\,h_{j-i+\lambda_i}(1,r)\,] = \frac{a_{\lambda+\delta}(x_1,x_2,\cdots,x_r)}{a_\delta(x_1,x_2,\cdots,x_r)} \tag{4.1}$$

where the skew-symmetric function is defined by

$$a_{(n_1,n_2,\cdots,n_r)}(x_1,x_2,\cdots,x_r) = \det_{r\times r}[\,x_j^{n_i}\,] \tag{4.2}$$

Further, there is a beautiful identity on the summation of Schur functions.

$$\sum_{\lambda} S_\lambda(x_1,x_2,\cdots,x_r) = \prod_{1\le k\le r}(1-x_k)^{-1}\prod_{1\le i<j\le r}(1-x_ix_j)^{-1} \tag{4.3}$$

where the summation runs over all partitions with at most r parts.

First we let $\nu = 0, t_i = 1$ and $b_j = n$ in (2.7). Then $S_{\lambda/\widetilde{\nu}}(t;b)$ reduces to the Schur function $S_\lambda(x) = \det_{r\times r}[\,h_{j-i+\lambda_j}(1,n)\,]$ which enumerates the column-strict reverse plane partitions of shape λ with all parts $\le n$. From (4.1) it is easy to deduce the following

Proposition 4.1.

$$S_\lambda(q,q^2,\cdots,q^n) = q^{|\lambda|+n(\lambda)}\prod_{x\leftarrow\lambda}\frac{\langle n+c(x)\rangle}{\langle h(x)\rangle} \tag{4.4}$$

where $\langle x\rangle = 1-q^x$ and other notations are as in [9, p. 27].

There are extensive literatures concerned with this topic. The interested readers may refer to [4–5], [7–10], [11–12] in which Remmel & Whitney have provided a lattice path proof in [11] recently.

Next, we put $\widetilde{\nu}_i = i, b_j = \infty$ and $\alpha_k = \lambda_k - k + 1$ in (2.7). Then the generating function for shifted reverse plane partitions of shape α is equal to

$$\begin{aligned} P_\alpha(q) &= \sum_{1\le t_1<t_2<\cdots<t_r}\left\{\det_{r\times r}[\,h_{\alpha_j-1}(t_i,a\infty)\,]\prod_{k=1}^{r}x_{t_k}\right\}_{x_i=q^i} \\ &= \sum_{1\le t_1<t_2<\cdots<t_r}\det_{r\times r}\left[\frac{q^{t_i\alpha_j}}{(q;q)_{\alpha_j-1}}\right] \end{aligned}$$

where $(x:q)_n = \prod_{k=1}^{n}(1-xq^{k-1})$. Define $|m| = \sum_{i=1}^{r} m_i$ for vector $m = (m_1,m_2,\cdots,m_r)$ and $n(m) = \sum_{k=1}^{r}(k-1)m_k$. Thus a simple replacement gives that

$$P_\alpha(q) = \frac{q^{|\alpha|}}{\prod_{1\le i\le r}(q:q)_{\alpha_i-1}}\sum_{\mu_1\ge\mu_2\ge\cdots\mu_r\ge0} a_{\mu+\delta}(q^{\alpha_r},q^{\alpha_{r-1}},\cdots,q^{\alpha_1})$$

By means of (4.3), this summation can be simplified as

Proposition 4.2.

$$P_\alpha(q) = \frac{q^{|\alpha|+n(\alpha)}}{\prod_{k=1}^{r}(q:q)_{\alpha_k}} \prod_{1\le i<j\le r} \frac{\langle \alpha_i - \alpha_j\rangle}{\langle \alpha_i + \alpha_j\rangle} \tag{4.5}$$

This result is consistent with Gansner's ([4, Corollary 7.2]) where he derived the result using the Hillman-Grassl correspondence.

A shifted reverse plane partition T of shape α is called a standard shifted tableau (SST) if the entries of T consist of just $1, 2, \cdots, |\alpha|$. According to the statements of Stanly ([13, Corollary 5.1–5.4]) the number of SST of shifted shape α is equal to (cf. Macdonald [9, p. 135])

$$\lim_{r\to 1} P_\alpha(q)(q;q)_{|\alpha|} = \binom{|\alpha|}{\alpha_1, \alpha_2, \cdots, \alpha_r} \prod_{1\le i<j\le r} \frac{\alpha_i - \alpha_j}{\alpha_i + \alpha_j} \tag{4.6}$$

where $\binom{n_1 + n_2 + \cdots + n_k}{n_1, n_2, \cdots, n_k}$ denotes the multinomial coefficient.

Finally, we set $\widetilde{\nu}_i = i$, $b_j = \infty$ and a symmetric partition $(\alpha_1 > \alpha_2 > \cdots > \alpha_r | \alpha_1 > \alpha_2 > \cdots > \alpha_r)$ in Frobenius' notation [1]. It follows from (2.7) that the generating function for symmetrical reverse plane partitions of shape $(\alpha|\alpha)$ is given by

$$Q_{(\alpha|\alpha)}(q) = \sum_{1\le t_1<t_2<\cdots<t_r} q^{-1n(\alpha)-\binom{r}{2}} \left\{ \det_{r\times r} [\, h_{\alpha_j}(t_i, \infty)\,] \right\} x_k = q^{2k}{}_q \sum_{k=1}^{r} t_k$$

where a simple transform has been made from the strict partitions to the ordinary ones. Then it is obvious that

$$Q_{(\alpha|\alpha)}(q) = q^{-2n(\alpha)-\binom{r}{2}} \sum_{0<t_1<t_2<\cdots<t_r} \det_{r\times r} \left[\frac{q^{(2\alpha_j+1)t_i}}{(q^2;q^2)_{\alpha_j}} \right]$$

By means of (4.1–4.3) this can be simplified as follows

$$\begin{aligned} Q_{(\alpha|\alpha)}(q) &= \frac{q^{2|\alpha|-2n(\alpha)+r-\binom{r}{2}}}{\prod_{k=1}^{r}(q^2;a^2)_{\alpha_k}} \sum_{\mu_1\ge\mu_2\ge\cdots\ge\mu_r\ge 0} a_{\mu+\delta}(q^{2\alpha_r+1}, q^{2\alpha_{r-1}+1}, \cdots, q^{2\alpha_1+1}) \\ &= q^{r+2|\alpha|} \prod_{k=1}^{r} \frac{(q;q^2)_{\alpha_i}}{(q;q)_{2\alpha_i+1}} \prod_{1\le i<j\le r} \frac{\langle 2(\alpha_i-\alpha_j)\rangle}{\langle 2(\alpha_i+\alpha_j+1)\rangle} \end{aligned}$$

This gives the following

Proposition 4.3. *Let $\lambda = (\alpha|\alpha)$ be a symmetric partition. Then we have*

$$Q_\lambda(q) = q^{|\lambda|} \prod_{k=1}^{r} \frac{(q;q^2)_{\alpha_k}}{(q;q)_{2\alpha_k+1}} \prod_{1\le i<j\le r} \frac{\langle 2(\alpha_i-\alpha_j)\rangle}{\langle 2(\alpha_i+\alpha_j+1)\rangle} \tag{4.7}$$

An alternative version of this statement in terms of hooks is obtained by Gansner ([4, Corollary 6.2]).

5. Concluded Remarks

On account of the fact that the concept of ladder tableaux is a natural extension to that of shifted tableaux, one can develop the related notions similar to the cases for ordinary tableaux.

From the ordinary and Gaussian binomial coefficients to elementary and complete symmetric functions, the "dominance method" described in section 2 and 3 may be viewed as a kind of formalization of the technique successfully used by Andrews [1] and Carlitz [3] to treat the counting problems of plane partitions. To make the presented technique carry into execution, two algebraic idenddes (1.1)–(1.2) play the vital role. From section 4 one can notice that the arguments demonstrated by dominance technique possess a nice flexibility in applications. A pair of intriguing open questions is whether there exist simple closed product expressions for the generating functions of shifted tableaux and symmetrical reverse plane partitions with the fixed shapes and the part-bounding.

References

1. G. E. Andrews, *The Theory of Partitions*, in "Encyclopedia of Mathematics and Applications: Vol. 2", Addision-Wesley, USA, 1976.
2. G. E. Andrews, *Plane Partitions (III): The weak Macdonanld conjecture*, Inv. Math. **53** (1979), 193–225.
3. L. Carlitz, *Rectangular arrays and plane partitions*, Acta Arithmetica **13** (1967), 29–47.
4. E. R. Gansner, *The Hillman-grassl correspondence and the enumeration of reverse plane partitions*, J. Combin. Theory (A) **30** (1981), 71–89.
5. I. P. Goulden & D. M. Jackson, "Combinatorial Enumeration", Wiley, New York, 1983.
6. A. P. Hillaman & R. M. Grassl, *Reverse plane partitions and tableau hook numbers*, J. Combin. Theory (A) **21:2** (1976), 216–221.
7. D. E. Knuth, *Permutations, matrices and generalized Young tableaux*, Pacific J. of Math **34** (1970), 709–727.
8. D. E. Littlewood, "The Theory of Group Characters", 2nd ed., Oxford University Press, Oxford, 1950.
9. I. G. Macdonald, "Symmetric Functions and Hall Polynomials", Clarendon Press, Oxford, 1979.
10. P. A. MacMahon, "Combinatory Analysis", Vol 1 & 2, Cambridge, 1915/1916.
11. J. B. Remmel & R. Whitney, *A bijective proof of the generating function for the number of reverse plane partitions via lattice paths*, Linear & Multilinear Algebra **16** (1984), 75–91.
12. R. P. Stanley, *Theory and application of plane partitions I & II*, Stud. in Appl. Math. **50** (1971), 167–188 & 259–279.

13. R. P. Stanley, *Ordered Structures and Partitions*, Amer. Math. Mem. no. **119** (1972).
14. B. Sagan, *An analog of Schensted's algorithm for shifted Young tableaux*, J. Combin. Theory (A) **27** (1979), 10–18.
15. M. L. Wachs, *Flagged schur functions, Schubert polynomials and symmetrizing operators*, J. Combin. Theory (A) **40** (1985), 276–289.