Some Combinatorial Inequalities on the Existence of Balanced Arrays

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Abstract. In this paper we drive some inequalities which the parameters of a two-symbol balanced array T (B-array) of strength four must satisfy for T to exist.

1. Introduction and preliminaries.

Definition: A balanced array (B-array) T of strength four, with two symbols, with m constraints and N treatment-combinations (runs) is a matrix T $(m \times N)$ with two symbols (say, 0 and 1) such that in every $(4 \times N)$ submatrix T^* of T, the following condition holds:

$$\lambda(\underline{\alpha}; T^*) = \lambda(P(\underline{\alpha}); T^*)$$

where $\lambda(\underline{\alpha}; T^*)$ denotes the frequency with which (4×1) vector $\underline{\alpha}$ appears in T^*, and $P(\underline{\alpha})$ is a vector obtained from $\underline{\alpha}$ by permuting its elements. If $\underline{\alpha}$ is a vector with i 1's $(i = 0, 1, 2, 3, 4)$ in it, then

$$\lambda(\underline{\alpha}; T^*) = \lambda(P(\underline{\alpha}); T^*) = \mu_i \quad \text{(say)}$$

and the vector $\underline{\mu}' = (\mu_0, \mu_1, \mu_2, \mu_3, \mu_4)$ is called the index set of the array T, and the B-array is sometimes denoted by $T\ (m, N, t = 4, s = 2; \underline{\mu}')$. The following result is quite obvious

$$N = \sum_{i=0}^{4} \binom{4}{i} \mu_i.$$

B-arrays have been found to be quite useful in the constructions of factorial designs, and tend to unite various branches of the combinatorial theory of design of experiments. For some literature on B-arrays in general and their usefulness, the interested reader may refer to the bibliography given at the end, and further references included therein.

2. Inequalities for the existence of balanced arrays.

It is easy to establish the following results.

Lemma 2. *A B-array T of strength four with index set* $\underline{\mu}' = (\mu_0, \mu_1, \mu_2, \mu_3, \mu_4)$ *is also of strength* $t' \leq 4$. *Considered as an array with* $t' = 3, 2,$ *and 1 the index sets* $\underline{\mu}^{t'}$ *are respectively given by* (A_0, A_1, A_2, A_3), (B_0, B_1, B_2), *and* (C_0, C_1) *where*

$$A_j = \sum_{i=0}^{1} \binom{1}{i} \mu_{i+j}, \quad B_j = \sum_{i=0}^{2} \binom{3}{i} \mu_{i+j},$$

and

$$C_j = \sum_{i=0}^{3} \binom{3}{i} \mu_{i+j}.$$

Lemma 2.2. *Let* $x_j (0 \leq j \leq m)$ *denote the number of columns with* j 1*'s in a B-array T with* $t = 4$. *Then the following must hold:*

$$\sum_{j=0}^{m} x_j = N \tag{2.1}$$

$$\sum_{j=0}^{m} j x_j = mC_1 \tag{2.2}$$

$$\sum j^2 x_j = m(m-1)B_2 + mC_1 \tag{2.3}$$

$$\sum j^3 x_j = m(m-1)(m-2)A_3 + 3m(m-1)B_2 + mC_1 \tag{2.4}$$

$$\begin{aligned} \sum j^4 x_j = {} & m(m-1)(m-2)(m-3)\mu_4 \\ & + 6m(m-1)(m-2)A_3 + 7m(m-1)B_2 + mC_1. \end{aligned} \tag{2.5}$$

Next we state two main results of this paper, briefly sketch their proofs, and illustrate their usefulness.

Theorem 2.1. *Consider a B-array T* $(m, t = 4, s = 2; \underline{\mu}')$. *Then we must have*

$$a m^3 + b m^2 + c m + d \geq 0 \tag{2.6}$$

where $a, b, c,$ *and* d *are polynomials of degree two in* μ_i*'s.*

Proof outline: Applying the following inequality to

$$\left(\sum j^2 x_j\right)^2 \leq \left(\sum j^4 x_j\right)\left(\sum x_j\right)$$

to (2.1), (2.3), (2.5) and after some simplification, we obtain (2.6), where

$$a = N\mu_4 - B_2^2,\ b = 6N\mu_3 - 2B_1B_2,\ c = N(7\mu_2 - 4\mu_3) - B_1^2,$$

and

$$d = N(\mu_1 - 4\mu_2 + \mu_3).$$

However, if we apply the inequality

$$\left(\sum j^3 x_j\right)^2 \le \left(\sum j^4 x_j\right)\left(\sum j^2 x_j\right)$$

to (2.2), (2.4), and (2.5) then we obtain, after some simplification, the following result:

Theorem 2.2. *For a B-array T with index set $\underline{\mu}'$ and $m \ge t = 4$ to exist, the following must hold*

$$a\,m^3 + b\,m^2 + c\,m + d \ge 0 \tag{2.7}$$

where $a = \mu_2\mu_4 - \mu_3^2$, $b = 5\mu_3^2 + \mu_3\mu_4 - 3\mu_2\mu_4 + \mu_1\mu_4$, $c = 4\mu_2(\mu_3 + \mu_4)$ $+4\mu_3(\mu_1 - \mu_3) - \mu_4(\mu_1 + \mu_3) - 2\mu_2^2$, *and* $d = 8\mu_2^2 + 2\mu_2(\mu_1 + \mu_3) - 4\mu_1\mu_3$.

Remark: The above two results (2.6), (2.7) are quite useful in checking the existence of a B-array T with a given m and $\underline{\mu}'$, and also in obtaining an upper bound on m for a given $\underline{\mu}'$.

Here we must stress that B-array T may not exist even if its parameters satisfy (2.6) and (2.7). However, we can be certain of the nonexistence of T if its parameters contradict either (2.6) or (2.7).

Example 1: Take $\underline{\mu}' = (1,3,3,1,0)$, and we apply (2.7) to it. Here $a = -1$, $b = 5$, $c = 2$, and $d = 84$. We have $-m^3 + 5m^2 + 2m + 84 \ge 0$.

It can be easily checked that the above inequality is contradicted at $m = 8$ for the first time. Thus $m \ne 8$, hence the largest value of m for which T with $\underline{\mu}' = (1,3,3,1,0)$ can possibly exist is $m = 7$.

Example 2: Consider T with $\underline{\mu}' = (1,2,5,2,1)$, and we check the application of (2.6). Here $a = -52$, $b = 296$, $c = 1100$, and $d = -768$. The polynomial inequality is $-52m^3 + 296m^2 + 1100m - 768 \ge 0$. This polynomial inequality is checked to be contradicted for $m = 9$ but not for $m = 8$. Thus, an upperbound on m is $= 8$.

Remark: Conditions (2.6) and (2.7) are necessary conditions, and a computer program can be easily prepared for checking (2.6) and (2.7) for any given list of B-arrays.

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