The Ramsey Numbers $R(K_3, K_8 - e)$ and $R(K_3, K_9 - e)$

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Abstract. We give a general construction of a triangle free graph on $4p$ points whose complement does not contain $K_{p+2} - e$ for $p \geq 4$. This implies that the Ramsey number $R(K_3, K_k - e) \geq 4k - 7$ for $k \geq 6$. We also present a cyclic triangle free graph on 30 points whose complement does not contain $K_9 - e$. The frst construction gives lower bounds equal to the exact values of the corresponding Ramsey numbers for $k = 6, 7$ and 8. The upper bounds are obtained by using computer algorithms. In particular, we obtain two new values of Ramsey numbers $R(K_3, K_8 - e) = 25$ and $R(K_3, K_9 - e) = 31$, the bounds $36 \leq R(K_3, K_{10} - e) \leq 39$, and the uniqueness of extremal graphs for Ramsey numbers $R(K_3, K_6 - e)$ and $R(K_3, K_7 - e)$.

1. Introduction and Notation

The two color Ramsey number $R(G, H)$ is the smallest integer n such that for any graph F on n vertices, either F contains G or the complement $\overline{F}$ contains H. In this paper we consider the case $G = K_3$ and $H = K_k - e$, the complete graph K_k minus an edge. Table I contains the values of some related Ramsey numbers. The entries of the first two rows are given by easy equalities $R(K_3 - e, K_k - e) = 2k - 3$ and $R(K_3 - e, K_k) = 2k - 1$, which can be derived by a straightforward reasoning. The value 21 of $R(K_3, K_k - e)$ for $k = 7$ was obtained by Grenda and Harborth in 1982 [5], where the authors list also all the values for $k \leq 6$. Recently, McKay and Zhang have calculated $R(K_3, K_8) = 28$ [7], other references for the classical case $R(K_3, K_k)$ can be found in [6], [7], [8], [9].

k								G	H
3	4	5	6	7	8	9	10		
3	5	7	9	11	13	15	17	$K_3 - e$	$K_k - e$
5	7	9	11	13	15	17	19	$K_3 - e$	K_k
5	7	11	17	21	25	31	36–39	K_3	$K_k - e$
6	9	14	18	23	28	36	40–43	K_3	K_k

Table I. Four related types Ramsey numbers $R(G, H)$

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All the graphs considered here are triangle free. Throughout this paper we adopt the following notation:

$\overline{G}$— complement of graph G
(G,H)-good graph F—graph F does not contain G and $\overline{F}$ does not contain H
(G,H,n)-good graph —(G,H)-good graph on n vertices
$K_p - e$—complete graph on p vertices without one edge
$G \equiv H$—graphs G and H are isomorphic
$e(G,H,n)$ —minimum number of edges in any (G,H,n)-good graph
$E(G,H,n)$ —maximum number of edges in any (G,H,n)-good graph
$G[S]$ —subgraphs of graph G induced by the set of vertices S
C_p—cycle of length p

2. Constructions

Construction 1: For $p \geq 1$, let $G_p = (V_p, E_p)$ be the graph on $4p$ vertices defined by:

$$V_p = \bigcup_{i=1}^{4} X_i, \quad \text{where } X_i = \{x_{in} : 1 \leq n \leq p\}, \text{ and}$$

$$E_p = \{\{x_{in}, x_{i+1,m}\} : i = 1,3, \quad 1 \leq n, m \leq p, \quad n \neq m\} \cup \{\{x_{in}, x_{jn}\} : i = 1,2,\ j = 3,4,\ 1 \leq n \leq p\}.$$

Observe that G_p is a regular graph of degree $p+1$ and that the induced graphs $G_p[X_1 \cup X_2]$ and $G_p[X_3 \cup X_4]$ are isomorphic to the complete bipartite graph $K_{p,p}$ with a 1-factor deleted. We say that vertex x_{in} is on *level* n. The set V_p is formed by p levels, each of them inducing a C_4 in G_p, in particular $G_1 \equiv C_4$. We leave for the reader, as an easy but interesting and time consuming exercise, to show that the graph G_4 on 16 vertices is isomorphic to the well known extremal graph related to the Ramsey number $R(3,3,3)$, which has vertices in $GF(16)$ and edges connecting points whose difference is a cube [4].

Theorem 1. *The graph G_p is a $(K_3, K_{p+2} - e, 4p)$-good graph for $p \geq 4$.*

Proof: One can easily verify that G_p has no triangles. Let S be any set of vertices, $S \subseteq V_p$, $|S| = p+2$. We will show that for $p \geq 4$ the induced graph $G_p[S]$ has at least two edges. If S has at least three vertices on the same level, then $G_p[S]$ has clearly at least two edges; otherwise S has at least two levels n and m with two vertices, say a and b on level n and c and d on level m. Since $p \geq 4$, S has at least two more vertices, u and v, on other levels. Suppose that $G_p[S]$ has at most one edge. Then without loss of generality we can assume that u is not connected to any vertex in $\{a,b,c,d\}$ and $u \in X_3$. Hence $\{a,b,c,d\} \subseteq X_1 \cup X_2 \cup X_3$ and one can easily check that $G_p[\{a,b,c,d\}]$ has at least two edges. ∎

Corollary 1. $R(K_3, K_k - e) \geq 4k - 7$ *for* $k \geq 6$.

Proof: Using Theorem 1, the lower bound is established by the graph G_{k-2}. ∎

Construction 2: Define graph $H = (\mathbf{Z}_{30}, E)$ by

$$E = \{\{i,j\} : i,j \in \mathbf{Z}_{30}, \quad i - j = \pm 1, \pm 3, \pm 9, \pm 14\}.$$

It is not very difficult, but again tedious, to check that the graph H is triangle free, has exactly 30 independent sets of size 8, namely the neighborhoods of vertices, and finally two different neighborhoods intersect in less than 7 points. Consequently the graph $\overline{H}$ does not contain $K_9 - e$, since the opposite would imply the existence of two independent sets of size 8 intersecting in seven points. Thus we can formulate the next Corollary.

Corollary 2. $R(K_3, K_9 - e) \geq 31$.

3. Enumerating small Graphs

In [8] the construction of a data base of all triangle free graphs with maximal independent set of size not larger than 5 was reported. This data base contains all $(K_3, K_k - e)$-good graphs for $k \leq 6$. These were extracted and the number of them is shown in the following tables for $k = 3, 4, 5$ and 6. A blank entry in a table denotes 0. Note that the values of $e(K_3, K_k - e, n)$ and $E(K_3, K_k - e, n)$ can be easily read by finding the location of the first and last nonzero entries in column n of the corresponding table. Observe also that G_4 is the unique $(K_3, K_6 - e, 16)$-good graph.

edges	number of vertices n				total
e	1	2	3	4	
0	1	1			2
1		1			1
2			1		1
3					0
4				1*a*	1
total	1	2	1	1	5

Table II. Number of $(K_3, K_3 - e)$-good graphs

The graphs contributing to the entries of Table II were constructed independently by hand. The correctness of the data in Tables III, IV and V was double checked by running extension algorithm used in the next section, i.e. the set of graphs obtained by extraction from the data base of (K_3, K_k)-good graphs was identical to the set of $(K_3, K_k - e)$-good graphs obtained by consecutive extensions followed by elimination of isomorphic copies of graphs. We also observe that column 10 of Table IV corresponds to Lemma 2 in [1], likewise the graph G_4 was also identified as a $(K_3, K_6 - e)$-good graph by Faudree, Rousseau and Schelp in [2] and it is represented by a 1 in column 16 of Table V. Finally we note a "curiosity" in column 10 of Table IV, namely the nonexistence of $(K_3, K_5 - e, 10)$-good graphs for $16 \le e \le 19$ edges. This is the first such hole known to the author (for additional data see [8], [9]).

In Tables II-VI some particular graphs of special interest have been marked as follows: a — square $K_{2,2}$, b — $K_{3,3}$, c — $K_{4,4}$, d — graphs from Lemma 2 in [1], e — Petersen graph, f — $K_{5,5}$, g — graph on $GF(16)$, $\{i,j\} \in E$ iff $i - j = x^3$, isomorphic to G_4, and h — unique $(K_3, K_7 - e, 20)$-good graph found by Grenda and Harborth in [5], isomorphic to G_5.

edges	number of vertices n						total
e	1	2	3	4	5	6	
0	1	1	1				3
1		1	1				2
2			1	2			3
3				2			2
4				1	2		3
5					2		2
6					1	1	2
7						1	1
8						1	1
9						1b	1
total	1	2	3	5	5	4	20

Table III. Number of $(K_3, K_4 - e)$-good graphs

4. Extensions

The system of algorithms with their implementations to construct all (K_3, K_k, n)-good graphs with e edges was described in [8] and used extensively in [9]. This technique requires the previous knowledge of all $(K_3, K_{k-1}, \overline{n})$-good graphs with $\overline{e}$ edges, for $\overline{n} < n$ and $\overline{e}$ ranging over the set of values, which can be determined by the method of Graver and Yackel [3]. The key to this method in our case is contained in the following Lemma.

Lemma 1 (variation of proposition 4 in Graver and Yackel [3] – 1968). *For any $(K_3, K_k - e, n)$-good graph G with e edges*

$$\Delta = ne - \sum_{i=0}^{k-1} n_i(e(K_3, K_{k-1} - e, n - i - 1) + i^2) \ge 0,$$

edges	number of vertices n										total
e	1	2	3	4	5	6	7	8	9	10	
0	1	1	1	1							4
1		1	1	1							3
2			1	2	2						5
3				2	3	1					6
4				1	4	4					9
5					2	7					9
6					1	7	5				13
7						4	8				12
8						2	12	2			16
9						1	8	5			14
10							1	14			16
11							1	12			13
12							1	10	1		12
13								4	1		5
14								2	3		5
15								1	1	1*de*	3
16								1*c*	1		2
17											0
18											0
19											0
20										1*d*	1
total	1	2	3	7	12	26	39	49	7	2	148

Table IV. Number of $(K_3, K_5 - e)$-good graphs

where n_i is the number of vertices of degree i in G, $n = \sum_{i=0}^{k-1} n_i$ and $2e = \sum_{i=0}^{k-1} i \cdot n_i$.

Lemma 1 gives reasonable lower bounds for $e(K_3, K_k - e, n)$ provided good lower bounds for $e(K_3, K_{k-1} - e, n - i - 1)$ are given. Furthermore, it permits the design of extension algorithms based on the ones used by Grinstead and Roberts in 1982 [6] to evaluate $R(3,9)$. Similarly as in [8], [9] we have implemented these algorithms for the case of $(K_3, K_k - e)$-good graphs and they have produced the results gathered in Tables VI and VII.

Let $e_k(n) = e(K_3, K_k - e, n)$ and let $N_k(n, e)$ be the number of nonisomorphic $(K_3, K_k - e, n)$-good graphs with e edges. Table VI presents all nonzero values of $e_7(n)$, and $N_7(n, e)$ for some values of n and e. Table VII contains similar data for $(K_3, K_8 - e, n)$-good graphs. In the case of $(K_3, K_7 - e, n)$-good graphs we have found all of them for $n \geq 18$: there are 225 such graphs for

edges	number of vertices n																
e	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	total
0	1	1	1	1	1												5
1		1	1	1	1												4
2			1	2	2	2											7
3				2	3	4	1										10
4				1	4	7	5	1									18
5					2	9	11	1									23
6					1	7	19	10									37
7						4	20	25	1								50
8						2	18	51	10								81
9						1	11	64	33								109
10							5	60	97	3							165
11							1	38	167	11							217
12							1	21	195	70							287
13								9	150	204							363
14								3	92	388	2						485
15								2	42	445	23						512
16								1	20	364	110						495
17									8	217	261						486
18									3	111	374	3					491
19									1	50	330	9					390
20									1	22	216	44					283
21										10	101	71					182
22										4	41	86					131
23										2	12	56					70
24										1	4	37	2				44
25										1f	1	22	5				29
26												13	8				21
27												6	5				11
28												2	2				4
29												1					1
30														2			2
31														2			2
32																	0
33																	0
34																	0
35															1		1
36																	0
37																	0
38																	0
39																	0
40																1g	1
total	1	2	3	7	14	36	92	286	820	1903	1475	350	22	4	1	1	5017

Table V. Number of (K_3,K_6-e)-good graphs

$n = 18$ with the number of edges ranging from 43 to 51, and unique graphs for $n = 19$ and 20. The graph G_5 is the unique $(K_3, K_7 - e, 20)$-good graph and obviously it is isomorphic to the graph defined by Grenda and Harborth in [5]. Also, there exist a unique $(K_3, K_7 - e, 19)$-good graph, which can be obtained from G_5 by the deletion of one vertex. The nonexistence of a $(K_3, K_8 - e, 25)$-good graphs implies, by Corollary 1, that $R(K_3, K_8 - e) = 25$. We note that G_6 has 84 edges, thus it is not a minimum graph. For further calculation of $R(K_3, K_9 - e)$ we need only the graphs in column $n = 22$ in Table VII and the values of $e_8(n)$ for $22 \leq n \leq 24$.

e	number of vertices n													
$N_7(n,e)$	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$e_7(n)$	2	3	4	5	8	11	15	19	24	30	37	43	54	60
$N_7(n,e)$	2	1	1	1	1	1	1	1	2	3	1	2	1	1h
$e_7(n)+1$				6	9	12	16	20	25	31	38	44		
$N_7(n,e)$		...		1	3	8	16	13	14	22	54	8		
$e_7(n)+2$									26	32	39	45		
$N_7(n,e)$						...			305	361	349	38		
$e_7(n)+3$										33	40	46		
$N_7(n,e)$									...	3251	1070	61		
$e_7(n)+4$												47		
$N_7(n,e)$											...	58		
$e_7(n)+5$												48		
$N_7(n,e)$												36		
$e_7(n)+6$												49		
$N_7(n,e)$												17		
$e_7(n)+7$												50		
$N_7(n,e)$												4		
$e_7(n)+8$												51		
$N_7(n,e)$												1		

Table VI. Number of $(K_3, K_7 - e, n)$-good graphs

Theorem 2. *$R(K_3, K_8 - e) = 25$ and $R(K_3, K_9 - e) = 31$.*

Proof: Corollaries 1 and 2 establish that 25 and 31 are lower bounds for $R(K_3, K_8 - e)$ and $R(K_3, K_9 - e)$, respectively. The fact that these values are also upper bounds follows from the calculations described above. For example, to prove $R(K_3, K_9 - e) \leq 31$ assume that G is a $(K_3, K_9 - e, 31)$-good graph with e edges. Then G can have vertices of degree 6, 7 and 8, and by Lemma 1 we have:

$$\Delta = 31e - (n_6(36+80) + n_7(49+70) + n_8(64+59)) = 31(e-116) - 3n_7 - 7n_8 \geq 0$$

There are three solutions in nonnegative integers for the latter, which are listed in Table VIII. One can easily conclude that G must be an extension of a $(K_3, K_8 - e, 22)$-good graph with 59 or 60 edges. There are 15 such graphs (see column 22 in Table VII). Running extension algorithm on these graphs did not produce G. Thus $R(K_3, K_9 - e) \leq 31$. ∎

e	number of vertices n					
$N_8(n,e)$	19	20	21	22	23	24
$e_8(n)$	37	44	51	59	70	80
$N_8(n,e)$	$\geq$20	$\geq$169	7	2	1	1
$e_8(n)+1$			52	60	71	81
$N_8(n,e)$			$\geq$375	13	2	0

Table VII. Number of $(K_3, K_8 - e, n)$-good graphs

n_6	n_7	n_8	e	Δ
0	0	31	124	31
1	0	30	123	7
0	2	29	123	8

Table VIII. Theorem 2

Using only Lemma 1 and Table VII we obtain:

$$e(K_3, K_9 - e, 30) \geq 111,$$
$$e(K_3, K_9 - e, 29) \geq 100, \quad \text{and}$$
$$e(K_3, K_9 - e, 28) \geq 90.$$

The latter inequalities and Lemma 1 imply the nonexistence of a $(K_3, K_{10} - e, 39)$-good graph, hence $R(K_3, K_{10} - e) \leq 39$. If we could prove $e(K_3, K_9 - e, 28) > 90$ then $R(K_3, K_{10} - e) \leq 38$. We have $36 = R(K_3, K_9) \leq R(K_3, K_{10} - e)$, so the lower bound also seems to be weak. There exists a good chance to calculate the exact value of $R(K_3, K_{10} - e)$! We conclude by stating the following Theorem.

Theorem 3. $36 \leq R(K_3, K_{10} - e) \leq 39$.

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