Nesting of cycle systems of even length

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Abstract. In this paper, we prove for any even intger $m \geq 4$ that there exists a nested m-cycle system of order n if and only if $n \equiv 1 \bmod 2m$, with at most 13 possible exceptions (for each value of m). The proof depends on the existence of certain group-divisible designs that are of independent interest. We show there is a group-divisible design having block sizes from the set $\{5,9,13,17,29,49\}$, and having u groups of size 4, for all $u \geq 5$, $u \not\equiv 7,8,12,14,18,19,23,24,33,34$.

1. Introduction

Let G be a graph, and let $m \geq 3$ be an integer. An *m-cycle decomposition* of G is an edge-decomposition of G into cycles of size m. We will write the m-cycle decomposition as a pair (G, C), where C is the set of cycles in the edge-decomposition. An m-cycle decomposition of K_n will be called an *m-cycle system of order n*. Of course, a 3-cycle system is a *Steiner triple system*; these designs exist for all orders $n \equiv 1$ or 3 modulo 6.

We will say that an m-cycle decomposition, (G, C), can be *nested* if we can associate with each cycle $C \in \mathrm{C}$ a vertex of G, which we denote $f(C)$, such that $f(C) \notin V(C)$, and such that the edges in $\{\{x, f(C)\} : x \in V(C), C \in \mathrm{C}\}$ form an edge-decomposition of G (where $V(C)$ denotes the vertex set of the cycle C). Alternatively, we can view a nested m-cycle decomposition as an edge-decomposition of the multigraph $2G$ into wheels with m spokes, where every edge occurs in one wheel as a spoke and in one wheel on the rim.

In this paper, we are interested in nested m-cycle systems for even values of m. It is easy to see that a necessary condition for the existence of a nested m-cycle system of order n is that $n \equiv 1 \bmod 2m$. The first examples of nested m-cycle systems to be studied in the literature were nested 3-cycle systems (i.e., nested Steiner triple systems). It was proven by Stinson [10] that there exists a nested Steiner triple system of order n if and only if $n \equiv 1$ modulo 6. More recently, Lindner, Rodger and Stinson [7] showed for each odd $m \geq 3$ that there exists a nested m-cycle system of order n if and only if $n \equiv 1$ modulo $2m$, with at most 13 possible exceptions.

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Much less is known regarding the existence of nested m-cycle systems for even values of m. In the smallest case, $m = 4$, it has been shown by Stinson [11] that the necessary condition $n \equiv 1 \bmod 8$ is sufficient for existence, with at most 6 possible exceptions. In this paper, we prove for any even $m \geq 4$ that there exists a nested m-cycle system of order n if and only if $n \equiv 1 \bmod 2m$, with at most 13 possible exceptions.

We prove the result when m is not a power of two in Section 2. For m a power of two, the proof is given in Section 3; the proof depends on the existence of certain group-divisible designs which are constructed in Section 4. The group-divisible designs we construct are of independent interest; we show there is a group-divisible design having block sizes from the set $\{5, 9, 13, 17, 29, 49\}$, and having u groups of size 4, for all but a few values of u (see Theorem 4.14).

2. Cycle lengths not a power of two

The construction we use for cycle lengths which are *not* a power of two depends on certain nested cycle decompositions of complete multipartite graphs. We will denote the complete multipartite graph having u parts of size t by $K_{(t^u)}$. Also, we refer to the parts as *holes*.

Lemma 2.1. *Suppose there is a nested m-cycle decomposition of $K_{(t^u)}$. Let $k \geq 1$. Then there is a nested (km)-cycle decomposition of $K_{((kt)^u)}$.*

Proof: Replace every vertex v of $K_{(t^u)}$ by k independent vertices, (named v_i, $1 \leq i \leq k$), thereby constructing $K_{((kt)^u)}$. Let $(K_{(t^u)}, \mathcal{C})$ be an m-cycle decomposition of $K_{(t^u)}$, and let f be a nesting of $\mathcal{C}$. Each cycle $C \in \mathcal{C}$ corresponds to a subgraph of $K_{((kt)^u)}$ isomorphic to the Cartesian product $C \otimes (K_k)^c$ (i.e. each vertex of C is replaced by k independent vertices, and each edge is replaced by k^2 edges forming a complete bipartite graph $K_{k,k}$). It is well-known that the graph $C \otimes (K_k)^c$ has an (mk)-cycle decomposition (this is a decomposition into Hamiltonian cycles; see [4] or [6]). The number of (mk)-cycles in this decomposition is k. Suppose these cycles are named C_i, $1 \leq i \leq k$. We define a nesting by associating with each C_i the vertex $f(C)_i$. If we do this for every cycle C, we obtain the desired nesting. ∎

We shall employ the following known class of nested m-cycle decompositions of $K_{((2m)^u)}$.

Lemma 2.2. *Suppose $m \geq 3$ is odd, $n = 2um + 1$, and $u \not\in \{1, 2, 3, 4, 6, 22, 23, 24, 26, 27, 28, 30, 34, 38\}$. Then there is a nested m-cycle decomposition of $K_{((2m)^u)}$.*

Proof: This follows from Theorem 2.2 and Theorem 3.1 of [7]. ∎

We also use the following class of nested cycle systems which are constructed by difference methods.

Lemma 2.3. *Suppose r is even. Then there is a nested r-cycle system of order $2r+1$.*

Proof: Let $r = 2k$, and define $\mathbf{a} = (a_1, \ldots, a_r)$ by

$$a_i = (-1)^i i, \quad \text{if } 1 \le i \le k-1$$
$$a_i = (-1)^{i+1} i, \quad \text{if } k \le i \le r,$$

where each a_i is reduced modulo $2r+1$. $\mathbf{a}$ represents the cycle $a_1 a_2 \cdots a_r a_1$. Let $\mathbf{C} = \{\mathbf{a} + j : j \in \mathbf{Z}_{2r+1}\}$. Then, it is easy to see that C is a cycle system of order $2n+1$. We define a nesting f of $\mathbf{C}$ by $f(\mathbf{a}+j) = j$, for every cycle $\mathbf{a} + j \in \mathbf{C}$. ∎

We now have the following immediate consequence.

Theorem 2.4. *Suppose $m \ge 3$ is odd, $n = 2um+1$, $u \notin \{2,3,4,6,22,23,24,26,27,28,30,34,38\}$, and $i \ge 0$. Then there exists a nested $(2^i m)$-cycle system of order $2^{i+1}um+1$.*

Proof: For $u = 1$, the result is given in Lemma 2.3. For $u > 1$, proceed as follows. Apply Lemma 2.1 to the m-cycle decompositions obtained from Lemma 2.2, using $k = 2^i$. We obtain a nested $(2^i m)$-cycle decomposition of $K_{((2^{i+1}m)^u)}$. Now, fill in the holes with nested $(2^i m)$-cycle systems of order $2^{i+1}m+1$ which exist by Lemma 2.3. ∎

Corollary 2.5. *Suppose r is even, r is not a power of 2, $n \equiv 1$ modulo $2r$, and $n \ge 78r+1$. Then there is a nested r-cycle system of order n.*

3. Cycle lengths a power of two

In this section we address the question of constructing nested 2^i-cycle systems. It was shown in [11] that the necessary condition $n \equiv 1 \bmod 8$ is sufficient for existence of a nested 4-cycle system, with at most 6 possible exceptions. Hence, we shall assume $i \ge 3$ for the remainder of this section.

Our construction for nested 2^i-cycle systems depends on the existence of certain group-divisible designs. A *group-divisible design* (or, GDD), is a triple $(X, \mathbf{G}, \mathbf{A})$, which satisfies the following properties:

1) $\mathbf{G}$ is a partition of X into subsets called *groups*,
2) $\mathbf{A}$ is a set of subsets of X (called *blocks*) such that any group and any block contain at most one common point, and
3) every pair of points from distinct groups occurs in a unique block.

The *group-type* of a GDD $(X, \mathbf{G}, \mathbf{A})$ is the multiset $\{|G| : G \in \mathbf{G}\}$. We usually use an "exponential" notation to describe group-types: a group-type $1^i 2^j 3^k \cdots$ denotes i occurrences of 1, j occurences of 2, etc. We will say that a GDD is a K-GDD if $|A| \in K$ for every $A \in \mathbf{A}$.

We shall prove the following result in Section 4.

Theorem 4.14. *Suppose* $u \geq 5$, $u \neq 7, 8, 12, 14, 18, 19, 23, 24, 33$, *or* 34. *Then there is a* $\{5, 9, 13, 17, 29, 49\}$*-GDD having group-type* 4^u.

The existence of certain nested 4-cycle decompositions will also prove useful.

Lemma 3.1 [11, Lemma 1]. *Suppose* $k \equiv 1$ *modulo* 4 *is a prime power. Then there is a nested 4-cycle decomposition of* $K_{(2^k)}$.

Combining Theorem 4.14 and Lemma 3.1, we have

Lemma 3.2. *Suppose* $u \geq 5$, $u \neq 7, 8, 12, 14, 18, 19, 23, 24, 33$, *or* 34. *Then there is a nested 4-cycle decomposition of* $K_{(8^u)}$.

Proof: Let $(X, \mathrm{G}, \mathrm{A})$ be a $\{5, 9, 13, 17, 29, 49\}$-GDD having group-type 4^u. Take 2 copies of every point x, say $\{x_1, x_2\}$. For every block A, construct a nested 4-cycle decomposition of $K_{(2^{|A|})}$ where the holes are $\{x_1, x_2\}$, $x \in A$. We get a nested 4-cycle decomposition of $K_{(8^{|X|/4})}$, where the holes are $\{x_1, x_2 : x \in G\}$, $G \in \mathrm{G}$. ∎

From this point on, we proceed as in Section 2.

Lemma 3.3. *Suppose* $u \geq 5$, $u \neq 7, 8, 12, 14, 18, 19, 23, 24, 33$, *or* 34, *and* $i \geq 3$. *Then there is a nested* (2^i)*-cycle decomposition of* $K_{((2^{i+1})^u)}$.

Proof: Apply Lemmata 3.2 and 2.1. ∎

Theorem 3.4. *Suppose* $u \geq 1$, $u \neq 2, 3, 4, 7, 8, 12, 14, 18, 19, 23, 24, 33$, *or* 34, *and* $i \geq 3$. *Then there is a nested* (2^i)*-cycle system of order* $2^{i+1}u + 1$.

Proof: For $u = 1$, apply Lemma 2.3. For $u \geq 2$, we proceed as follows. Construct a nested (2^i)-cycle decomposition of $K_{((2^{i+1})^u)}$, using Lemma 3.3, and then fill in the holes with nested (2^i)-cycle systems of order $2^{i+1} + 1$ which exist by Lemma 2.3. ∎

Corollary 3.5. *Suppose* $r \geq 4$ *is a power of two,* $n \equiv 1$ *modulo* $2r$, *and* $n \geq 70r + 1$. *Then there is a nested* r*-cycle system of order* n.

4. Group divisible designs with block sizes from $\{5, 9, 13, 17, 29, 49\}$

In this section, we prove Theorem 4.14. This theorem is an extension of results proved in Mullin, Schellenberg, Vanstone and Wallis [8] and is proved using the techniques developed in that paper. It will be useful to recall several results from [8], but first, we define some design-theoretic terminology.

A *pairwise balanced design* (or, PBD) is a pair (X, A), such that X is a set of elements (called *points*) and A is a set of subsets of X (called *blocks*), such that every unordered pair of points is contained in a unique block of A. If v is a positive integer and K is a set of positive integers, then we say that (X, A) is a

(v, K)-PBD if $|X| = v$, and $|A| \in K$ for every $A \in \mathbf{A}$. The integer v is called the *order* of the PBD.

Using this notation, we can define a $(v, k, 1)$-BIBD (*balanced incomplete block design*) to be a $(v, \{k\})$-PBD. A BIBD is *resolvable* if the set of blocks can be partitioned into parallel classes, each of which is a partition of the points.

For any set K of positive integers, define $\mathbf{B}(K) = \{v$: there is a (v, K)-PBD$\}$. We say that $\mathbf{B}(K)$ is the closure of K. K is said to be *PBD-closed* if $K = \mathbf{B}(K)$.

We are interested in designs (PBDs and GDDs) with block sizes 5, 9, 13, 17, 29, or 49; hence we define $K_5 = \{5, 9, 13, 17, 29, 49\}$. The results in [8] pertain mostly to designs with block sizes from $\{5, 9, 13\}$. However, all the results from [8] which we use remain true when the set of block sizes is enlarged to include 17, 29, and 49, so we will restate them in this form. The following PBD result was proved in [8] and [5].

Theorem 4.1. *Suppose* $v \equiv 1$ *modulo* 4, $v \not\equiv 33, 57, 93, 133$. *Then there is a* (v, K_5)*-PBD.*

Hence, $v \in \mathbf{B}(K_5)$ for all such v.

We construct our GDDs recursively, using the following construction of Wilson [14].

Fundamental GDD Construction: Let $(X, \mathbf{G}, \mathbf{A})$ be a GDD, and let $s : X \to \mathbf{Z}^+ \cup \{0\}$ be a function. For every block $A \in \mathbf{A}$, suppose that we have a K-GDD of type $\{s(x) : x \in A\}$. Then there exists a K-GDD of type $\{\sum_{x \in G} s(x) : G \in \mathbf{G}\}$.

We shall refer to the Fundamental Construction as FC.

Define $U = \{u$: there exists a K_5-GDD of group-type $4^u\}$. Our goal is to show that all positive integers are in the set U, with a few exceptions. Our main construction is from [8]; it uses transversal designs, which we now define. A *transversal design* TD(k, n) can be defined to be a $\{k\}$-GDD of group-type n^k. It is well-known that a TD(k, n) is equivalent to $k - 2$ mutually orthogonal Latin squares of order n.

Lemma 4.2 [8, Corollary 4.6]. *Suppose there is a* $TD(6, m)$, $0 \le t \le m$, $\{m, t\} \cap \{8, 14, 23, 33\} = \emptyset$, *and* $\{m, t\} \cap U \ne \emptyset$. *Then,* $5m + t \in U$.

Proof: Truncate $m - t$ points from a group of the TD$(6, m)$, producing a (5, 6)-GDD of group-type $m^5 t^1$. Give every point weight 4 and apply FC, filling in $\{5\}$-GDDs of group-types 4^5 and 4^6 (these are constructed by deleting a point from a $(21, 5, 1)$-BIBD and a $(25, 5, 1)$-BIBD, respectively). This produces a $\{5\}$-GDD of group-type $(4m)^5(4t)^1$. Suppose $t \in U$ (the case $m \in U$ is handled similarly) and let G_0 be the group of size $4t$. Adjoin a new point ∞ to each group, producing a $(4(5m + t) + 1, \mathbf{B}(K_5))$-PBD. Now, delete some point $x \in G_0$ from every block in which it occurs in the design. This produces a

$B(K_5)$-GDD of group-type $4^{5m}(4t)^1$. Replace every block B by a $(|B|, K_5)$-PBD, constructing a K_5-GDD of group-type $4^{5m}(4t)^1$. Finally, replace the group of size $4t$ by a K_5-GDD of group-type 4^t. This yields a K_5-GDD of group-type 4^{5m+t}, as desired. ∎

TD$(6, m)$ are known to exist as follows.

Theorem 4.3. *Suppose* $m \neq 2, 3, 4, 6, 10, 14, 18, 22, 26, 30, 34$, *or* 42. *Then there exists a TD*$(6, m)$.

Proof: For most values of m, this result is proved in [2]. A few unknown cases have recently been constructed as follows. A TD(6, 24) was produced by Roth and Peters [9]; a TD(6, 20) was found by Todorov [12]; TD(6, 28) and TD(6, 52) were constructed by Abel [1]; and TD(6, 38) and TD(6, 44) have been constructed by Todorov [13]. ∎

Lemma 4.4. *Suppose* $u \equiv 1$ *modulo* 4, $u \neq 33, 57, 93, 133$. *Then* $u \in U$.

Proof: There exists a resolvable $(3u+1, 4, 1)$-BIBD, by [3]. Adjoin a new point to each of the u parallel classes, and adjoin the blocks of a (u, K_5)-PBD on the new points. Now, delete an old point, thus forming the desired GDD. ∎

Lemma 4.5 [8, Lemma 6.2]. *Suppose* $u \equiv 0$ *or* 1 *modulo* 5. *Then* $u \in U$.

We can now eliminate three of the four exceptions in Lemma 4.4.

Lemma 4.6. *Suppose* $u = 57, 93$, *or* 133. *Then* $u \in U$.

Proof: Apply Lemma 4.2, noting that $57 = 5 \times 11 + 2$, $93 = 5 \times 16 + 13$, and $133 = 5 \times 24 + 13$; and that $\{11, 13\} \subseteq U$. ∎

As a result of Lemmata 4.4–4.6, we have that $u \in U$ if $u \equiv 0, 1, 5, 6, 9, 10, 11, 13, 15, 16$, or 17 modulo 20, $u \geq 5$, $u \neq 33$. For the remaining 9 congruence classes modulo 20, we can already establish preliminary bounds beyond which $u \in U$. These bounds are all applications of Lemma 4.2, using TD$(6, m)$ from Theorem 4.3.

The following lemmata will be useful in handling some special cases.

Lemma 4.7 [8, Lemma 6.3]. *Suppose* $u \equiv 2$ *or* 22 *modulo* 25, $u > 2$. *Then* $u \in U$.

Lemma 4.8. *Suppose there is a TD*(r, m), $0 \leq r \leq m$, *and* $\{r, m\} \subseteq U$. *Then* $rm \in U$.

Proof: The TD yields an $(rm, \{r, m\})$-PBD. The result follows since U is PBD-closed ([8, Lemma 6.1]). ∎

Table 1

u modulo 20	equation	allowable values of m	values of u handled
$u \equiv 2 \bmod 20$	$u = 5n + 17$	$m \equiv 1 \bmod 4$, $m \geq 17, m \neq 33$	$u \geq 102, u \neq 182$
$u \equiv 3 \bmod 20$	$u = 5m + 13$	$m \equiv 2 \bmod 4$, $m \geq 46$	$u \geq 243$
$u \equiv 4 \bmod 20$	$u = 5m + 9$	$m \equiv 3 \bmod 4$, $m \geq 11, m \neq 23$	$u \geq 64, u \neq 124$
$u \equiv 7 \bmod 20$	$u = 5m + 17$	$m \equiv 2 \bmod 4$, $m \geq 46$	$u \geq 247$
$u \equiv 8 \bmod 20$	$u = 5m + 13$	$m \equiv 3 \bmod 4$, $m \geq 15, m \neq 23$	$u \geq 88, u \neq 128$
$u \equiv 12 \bmod 20$	$u = 5m + 17$	$m \equiv 3 \bmod 4$, $m \geq 19, m \neq 23$	$u \geq 112, u \neq 132$
$u \equiv 14 \bmod 20$	$u = 5m + 9$	$m \equiv 1 \bmod 4$, $m \geq 9, m \neq 33$	$u \geq 54, u \neq 174$
$u \equiv 18 \bmod 20$	$u = 5m + 13$	$m \equiv 1 \bmod 4$, $m \geq 17, m \neq 33$	$u \geq 98, u \neq 178$
$u \equiv 19 \bmod 20$	$u = 5m + 9$	$m \equiv 2 \bmod 4$, $m \geq 46$	$u \geq 239$

Lemma 4.9. $32 \in U$.

Proof: A $(129, \{5, 29\})$-PBD having a unique block of size 29 is presented in [5]. Delete some point *not* in the block of size 29 to construct a $(5, 29)$-GDD of type 4^{32}. ∎

The following is a variation of Lemma 4.2.

Lemma 4.10 **[8, Corollary 5.15].** *Suppose there is a* $TD(6, m)$, $0 \leq t \leq m$, *and* $b \geq 0$. *Suppose there is a* $(4m + b, K_5)$*-PBD, say* $(Y, \mathbf{B})$, *which contains a block* B *of size* b, *and suppose there is a* $(4t + b, K_5)$*-PBD. Then there is a* $(20m + 4t + b, K_5)$*-PBD. If, further, there is a point* $x \in Y \setminus B$ *which occurs only in blocks of size* 5, *then* $5m + t + ((b-1)/4) \in U$.

Proof: As in Lemma 4.2, construct a $\{5\}$-GDD of group-type $(4m)^5(4t)^1$, say $(X, \mathbf{G}, \mathbf{A})$. Let $\Omega \cap X = \emptyset$, $|\Omega| = b$. For each group G of size $4m$, let $(G \cup \Omega, \mathbf{B}_G)$ be a $(4m + b, K_5)$-PBD, where $\Omega \in \mathbf{B}_G$ is a block of size b. For the group G_0 of size $4t$, let $(G_0 \cup \Omega, \mathbf{B}_0)$ be a $(4t+b, K_5)$-PBD. Then $(X \cup \Omega, \cup_{G \in \mathrm{G}} (\mathbf{B}_G \backslash \{\Omega\}) \cup \mathbf{B}_0 \cup \mathrm{A})$ is a $(20m+4t+b, K_5)$-PBD. If we delete a point x as hypothesized above, then we obtain a K_5-GDD of group-type $4^{5m+t+((b-1)/4)}$, as desired. ∎

Corollary 4.11. $39 \in U$.

Proof: Apply Lemma 4.10 with $m = 7, t = 2, b = 9$, so $4m + b = 37$ and $4t + b = 17$. By adjoining infinite points to the parallel classes of a resolvable $(28, 4, 1)$-BIBD, we can construct a $(37, \{5, 9\})$-PBD which contains a (unique) block of size 9. A block of size 17 is a $(17, \{17\})$-PBD. We obtain a K_5-GDD of group-type 4^{39}, as desired. ∎

Corollary 4.12. $42 \in U$.

Table 2

u	is $u \in U$?	authority	construction
7	no		
8	no		
12	?		
14	?		
18	?		
19	?		
22	yes	Lemma 4.7	$u \equiv 22$ modulo 25
23	?		
24	?		
27	yes	Lemma 4.7	$u \equiv 2$ modulo 25
28	yes	Lemma 4.2	$5 \times 5 + 3$
32	yes	Lemma 4.9	
34	?		
38	yes	[8, Table 1]	
39	yes	Corollary 4.11	
42	yes	Corollary 4.12	
43	yes	[8, Table 1]	
44	yes	[8, Table 1]	
47	yes	Lemma 4.7	$u \equiv 22$ mod 25
48	yes	Lemma 4.2	$5 \times 9 + 3$
52	yes	Lemma 4.7	$u \equiv 2$ mod 25
58	yes	Lemma 4.2	$5 \times 11 + 3$
59	yes	Lemma 4.2	$5 \times 11 + 4$
62	yes	[8, Table 1]	
63	yes	Corollary 4.13	

u	is $u \in U$?	authority	construction
67	yes	Lemma 4.2	$5 \times 13 + 2$
68	yes	Lemma 4.2	$5 \times 13 + 3$
72	yes	Lemma 4.7	$u \equiv 22 \bmod 25$
78	yes	Lemma 4.2	$5 \times 15 + 3$
79	yes	Lemma 4.2	$5 \times 15 + 4$
82	yes	Lemma 4.2	$5 \times 16 + 2$
83	yes	Lemma 4.2	$5 \times 16 + 3$
87	yes	Lemma 4.2	$5 \times 16 + 7$
92	yes	Lemma 4.2	$5 \times 17 + 7$
99	yes	Lemma 4.8	9×11
103	yes	Lemma 4.2	$5 \times 20 + 3$
107	yes	Lemma 4.2	$5 \times 20 + 7$
119	yes	Lemma 4.2	$5 \times 20 + 19$
123	yes	Lemma 4.2	$5 \times 21 + 18$
124	yes	Lemma 4.2	$5 \times 21 + 19$
127	yes	Lemma 4.2	$5 \times 25 + 2$
128	yes	Lemma 4.2	$5 \times 25 + 3$
132	yes	Lemma 4.2	$5 \times 25 + 7$
139	yes	Lemma 4.2	$5 \times 27 + 4$
143	yes	Lemma 4.2	$5 \times 25 + 18$
147	yes	Lemma 4.2	$5 \times 29 + 2$
159	yes	Lemma 4.2	$5 \times 31 + 4$
163	yes	Lemma 4.2	$5 \times 32 + 3$
167	yes	Lemma 4.2	$5 \times 32 + 7$
174	yes	Lemma 4.2	$5 \times 31 + 19$
178	yes	Lemma 4.2	$5 \times 35 + 3$
179	yes	Lemma 4.2	$5 \times 35 + 4$
182	yes	Lemma 4.2	$5 \times 36 + 2$
183	yes	Lemma 4.2	$5 \times 36 + 3$
187	yes	Lemma 4.2	$5 \times 37 + 2$
199	yes	Lemma 4.2	$5 \times 39 + 4$
203	yes	Lemma 4.2	$5 \times 40 + 3$
207	yes	Lemma 4.2	$5 \times 41 + 2$
219	yes	Lemma 4.2	$5 \times 43 + 4$
223	yes	Lemma 4.2	$5 \times 41 + 18$
227	yes	Lemma 4.2	$5 \times 45 + 2$

Proof: Apply Lemma 4.10 with $m = 8, t = 1, b = 5$, so $4m + b = 37$ and $4t + b = 9$. As in Corollary 4.11, we can construct a $(37, \{5, 9\})$-PBD which contains a block of size 5. A block of size 9 is a $(9,\{9\})$-PBD. We obtain a K_5-GDD of group-type 4^{42}, as desired. ∎

Corollary 4.13. $63 \in U$.

Proof: Apply Lemma 4.10 with $m = 12, t = 2, b = 5$, so $4m + b = 53$ and $4t + b = 13$. By adjoining infinite points to the parallel classes of a resolvable $(40,4,1)$-BIBD, we can construct a $(53,\{5,13\})$-PBD which contains a block of size 5. A block of size 13 is a $(13,\{13\})$-PBD. We obtain a K_5-GDD of group-type 4^{63}, as desired. ∎

In Table 2, we list all values of $u \equiv 2,3,4,7,8,12,14,18$, or 19 modulo 29, $u \geq 7$, which are not handled in Table 1. For each such u, we indicate if it is known that $u \in U$. If so, we give a construction to show that $u \in U$.

Summarizing the results of Lemmata 4.4–4.6, Table 1, and Table 2, we have our existence result.

Theorem 4.14. *Suppose* $u \geq 5, u \not\equiv 7,8,12,14,18,19,23,24,33,34$. *Then there is a* $\{5,9,13,17,29,49\}$*-GDD having group-type* 4^u.

References

1. R.J.R. Abel, *Four mutually orthogonal Latin squares of orders 28 and 52*, preprint.
2. Th. Beth, D. Jungnickel and H. Lenz, "Design Theory", Bibliographisches Institut, Zurich, 1985.
3. H. Hanani, D. K. Ray-Chaudhuri and R. M. Wilson, *On resolvable designs*, Discrete Math. 3 (1972), 343–357.
4. G. Hetyei, *On Hamiltonian circuits and 1-factors of regular complete n-partite graphs*, (in Hungarian), Acta Acad. Pedagog. Civitate Pecs, Ser. 6, Math. Phys. Chem. Tech. 19 (1975), 5–10.
5. E. R. Lamken, W. H. Mills and R. M. Wilson, *A pairwise balanced design*, preprint.
6. R. Laskar, *Decomposition of some composite graphs into Hamiltonian cycles*, in "Proc. Fifth Hungar. Colloq.", North-Holland, Amsterdam, 1978, pp. 705–716.
7. C. C. Lindner, C. A. Rodger and D. R. Stinson, *Nesting of cycle systems of odd length*, Discrete Math. 77 (1989), 191–203.
8. R. C. Mullin, P. J. Schellenberg, S. A Vanstone and W. D. Wallis, *On the existence of frames*, Discrete Math. 37 (1981), 79–104.
9. R. Roth and M. Peters, *Four pairwise orthogonal Latin squares of order 24*, J. Combin. Theory A 44 (1987), 152–155.
10. D. R. Stinson, *The spectrum of nested Steiner triple systems*, Graphs and Combinatorics 1 (1985), 189–191.

11. D. R. Stinson, *On the spectrum of nested 4-cycle systems*, Utilitas Math. **33** (1988), 47–50.
12. D. T. Todorov, *Four mutually orthogonal Latin squares of order 20*, Ars Combinatoria **17** (1989), 63–65.
13. D. T. Todorov, *unpublished*, (private communication from P. J. Schellenberg).
14. R. M. Wilson, *Constructions and users of pairwise balanced designs*, Math. Centre Tracts **55** (1974), 19–42.