Slightly Improved Upper Bounds for Equidistant Permutation Arrays of Index One

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This note will closely follow the terminology and results in Mathon [1]. In case 2 on page 305, it states that $s_1 + \cdots + s_r$ is maximum if each row of S contains $r-2$ blocks of size $r-3$, one block of size $r-x$ and one empty block. But there are no blocks of size smaller than $r-x+2$. If there were then an element in that block would occur with only $(r-1)(r-4)+r-x = r^2-4r+4-x$. This implies $r^2-4r+5-x$ elements in the design which is a contradiction. Hence the way to maximize the number of pairs in a row consists of $(r-3)$ blocks of size $r-3$, 1 block of size $(r-5)$ and 1 block of size $(r-x+2)$. This gives $\frac{1}{2}(r^3-8r^2+25r-4-2rx+x^2-3x)$ pairs. Multiply this by r to give the maximum number of pairs in the GRS. This must be larger than $\binom{r^2-4r+6-x}{2}$ and gives $x \geq \frac{11}{2}+\sqrt{2r-\frac{31}{4}-\frac{8}{r-1}}$ or $v \leq r^2-4r+\frac{1}{2}-\sqrt{2r-\frac{31}{4}-\frac{8}{r-1}}$. For this to work $r-x$ must be greater than 0.

Similarly for case 1, the smallest possible block must be $r-x+2$ and for this to work $r-x-1>1$ or $r-x>2$. The row with the block of size $r-2$ has 3 more pairs than the other rows so the number of pairs in S is 3 more than in the previous case giving

$$x \geq \frac{11}{2}+\left(2r-\frac{31}{4}-\frac{14}{r-1}\right)^{1/2}$$

$$\text{or } v \leq r^2-4r+\frac{1}{2}+\left(2r-\frac{31}{4}-\frac{14}{r-1}\right)^{1/2}.$$

For $v=43$, $r=9$, $x=8$ the argument fails but some *ad hoc* arguments prove that the row containing the big block has blocks of sizes 7, six 6's, and two zeros. The other rows are like case 1 giving a total of 903 pairs which is exactly correct. Every element not in the big block occurs only in blocks of size 6 or 4. Every element in the big block occurs in blocks of size 6 and two blocks of size 3. Hence there are $3\times 8 = 24$ elements in blocks of size 3 but only $7\times 2 = 14$ elements counted the other way. Hence v must be less than 43 when $r=9$. This can be summarized as follows:

Theorem. *If* $s(r, v)$ *has side* $r \geq 9$, *then*

$$v \leq \left\lfloor r^2 - 4r + \frac{1}{2} + \left(2r - \frac{31}{4} - \frac{14}{r-1}\right)^{1/2} \right\rfloor .$$

Of course the improvement is only 2 or 3 over Mathon's result. We can update his table as follows.

r	8	9	10	11	12	13
L(r)	16	18	27	24	35	40
U(r)	31	42	57	73	92	113

The result $L(13) = 40$ was discovered by Mathon [2] by finding a doubly resolvable BIBD $(40, 130, 13, 4, 1)$.

References

1. R. Mathon, *Bounds for Equidistant Permutation Arrays of Index One*, Combinatorial Mathematics: Proceedings of the Third International Conference, Vol. 555, Annals of the N.Y. Acad. Sciences, pp 303-309.
2. R. Mathon, *private communication.*