The Range of the Determinant Function on the Set of $n \times n$ $(0,1)$-Matrices

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Abstract. As stated in [2], there is a conjecture that the determinant function maps the set of $n \times n$ $(0,1)$-matrices onto a set of consecutive integers for any given n. While this is true for $n \leq 6$, it is shown to be false for $n = 7$. In particular there is no 7×7 determinant in the range $28 - 31$ but there is one equal to 32. Then the more general question of the range of the determinant function for all n is discussed. A lower bound is given on the largest string of consecutive integers centered at 0, each of which is a determinant of an $n \times n$ $(0,1)$-matrix.

1. Introduction.

Let us begin with a few conventions:

- Write $\mathcal{T}_n$ for the set of $n \times n$ $(0,1)$-matrices.
- $\beta_n = \max_{M \in \mathcal{T}_n} |M|$.
- $D_k(n)$ is the assertion that there is a matrix in $\mathcal{T}_n$ whose determinant is k. Abusing the notation somewhat, any such matrix shall be denoted a $D_k(n)$.

Knowing the range of the determinant function on $\mathcal{T}_n$ may be useful, since the existence of a combinatorial design is equivalent to the existence of the corresponding $(0,1)$-incidence matrix, and hence implies that the corresponding (usually predictable, as in the case of SBIBD's) determinant exists when that matrix is square.

Brenner and Cummings ([2], 1972) relate the following conjecture: that for a given n, the determinant function maps $\mathcal{T}_n$ onto a set of consecutive integers. In my notation, this translates to "$|k| \leq \beta_n$ implies $D_k(n)$" for $n > 1$ (since we have $D_k(n)$ if and only if $D_{-k}(n)$ — see Lemma 3.1 — and consequently this set is symmetric with respect to 0). Subsequently we refer to this as the *Consecutive Integer Determinant Conjecture.*

Brenner says that he had raised the question with Marshall Hall in correspondence before 1972. He also relates that some time ago he heard someone had a counterexample. In any case I have not been able to find any that appear in print to date.

In Section 2 we show that the above conjecture is in fact false. Section 3 introduces a lemma which can be used to show that certain determinants exist. In the last section we outline some of the relevant unsolved questions, indicating some partial results obtained.

A final resolution to some of these questions may lead to a solution of the *Hadamard maximum determinant problem*, which is the question as to what is the maximum determinant among (± 1)-matrices of a given order ([2], [8]).[1] This is because $D_k(n)$ is equivalent to the existence of a (± 1)-matrix of order $n+1$ with determinant $2^n k$, modulo the following transformation:

$$2^n|D| = |2D| = \left|\begin{array}{c|ccc} 1 & 1 & \cdots & 1 \\ \hline 0 & & & \\ \vdots & & 2D & \\ 0 & & & \end{array}\right| = \left|\begin{array}{c|ccc} 1 & 1 & \cdots & 1 \\ \hline -1 & & & \\ \vdots & & 2D-J & \\ -1 & & & \end{array}\right|, \qquad (1)$$

where D is a $D_k(n)$ [11].[2]

Let us take note of a nice (well-known) property of rank-one matrices which will prove useful at several points. A rank-one matrix U is similar to an upper-triangular matrix. The only possible form for an upper-triangular rank-one matrix with trace equal to that of U is

$$\left(\begin{array}{c|c} trU & * \\ \hline 0 & \\ \vdots & 0 \\ 0 & \end{array}\right). \qquad (2)$$

Noting that trace is similarity-invariant, immediately we have:

Lemma 1.1 (the "Rank-one Lemma").
If U is rank-one, then $|I+U| = 1 + tr(U)$.

2. The consecutive integer determinant conjecture is false.

We shall see in the next section that the consecutive integer determinant conjecture holds for $n = 1, \ldots, 6$. However, that is as far as it goes, as the following result shows:

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[1] the *Hadamard bound*, $n^{\frac{n}{2}}$, holds in all orders. A (± 1)-matrix achieving this bound is called a *Hadamard matrix*. Equivalently, a Hadamard matrix H is one which satisfies $HH^t = nI$.

[2] and using the fact that we can negate rows and columns of a matrix without changing the absolute value of its determinant.

Theorem 2.1. *There is no $D_k(7)$ for $27 < k < 32$.*

Proof: First note that $\beta_6 = 9$ (see Section 3). This tells us that any matrix whose minors along a row are in $\mathcal{T}_6$, and whose entries in this row are from $\{0, \pm 1\}$, has determinant less than or equal to 9 times the number of non-zero entries in this row. We shall use this observation repeatedly in the proof.

Now suppose A is a $D_k(7)$ with $k > 27$. Write $\mathbf{r}_1, \ldots, \mathbf{r}_7$ for the rows of A, r_i for $\|\mathbf{r}_i\|$, and $\lambda_{i,j}$ for $\langle \mathbf{r}_i, \mathbf{r}_j \rangle$, $i \neq j$. Then proceed as follows:[3]

1. For each i, $r_i \geq 4$. This follows from the above observation.
2. Suppose $r_1 = 7$. Then:

$$A = \begin{matrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \\ \end{matrix} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & * & * & * \\ & & & \vdots & & & \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 0 & 0 & * & * & * \\ & & & \vdots & & & \end{pmatrix}. \quad (3)$$

   Here $\sim$ denotes equivalence via the action of adding a multiple of one row to another.[4] Plainly we have forced $|A| \leq 27$, which is a contradiction.

   We argue similarly for each row, concluding that for each i, $r_i < 7$.

3. Suppose that $r_2 = r_3 = \ldots = r_7 = 4$. Then for $i, j > 1$ we cannot have $\lambda_{i,j} = 3$ or 4 — else $|A| \leq 18$ as in (3). Thus $\lambda_{i,j} = 1$ or 2.

   Now suppose $\lambda_{2,3} = 1$. Then:

$$\begin{matrix} \mathbf{r}_2 \\ \mathbf{r}_3 \end{matrix} \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} \quad (4)$$

   and we cannot have $\lambda_{2,j} = 1$ for $j > 3$, else then $\lambda_{3,j} = 3$ or 4, which, as we have just seen, cannot happen. Similarly, $\lambda_{3,j} \neq 1$, and so we have:

$$\begin{matrix} \mathbf{r}_2 \\ \mathbf{r}_3 \\ \mathbf{r}_j \end{matrix} \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}. \quad (5)$$

   Clearly, the first column of A must then have at most three nonzero entries. Expanding by minors along this column, we see that $|A| \leq 27$, which is a contradiction.

   We conclude that if all $r_i = 4$ with at most one exception, then $\lambda_{i,j} = 2$ when $r_i = r_j = 4$.

4. Suppose that $r_1 = 6$. If in addition r_2 has a 0 in the same column as the 0 in $\mathbf{r}_1$, then:

$$A = \begin{matrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \\ \end{matrix} \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & * & * \\ & & & \vdots & & & \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & * & * \\ & & & \vdots & & & \end{pmatrix}. \quad (6)$$

[3] With no loss of generality, we assume at each step that the columns of A are arranged conveniently for display purposes.

[4] a determinant-preserving operation.

This gives a contradiction as before. Thus, $\mathbf{r}_j$ must have a 1 in this column for $j > 1$.

It follows in the same manner that $r_j = 4$ for $j > 1$. Now by step 3, $\lambda_{i,j} = 2$ for $i, j > 1$. Letting B be the submatrix obtained by excluding the first row and column of A, then, this all translates to $BJ = B^t J = 3J$ and $BB^t = 2I + J$. Thus $BB^t J = 9J = 2IJ + J^2 = 8J$, which is a contradiction (thus showing the well-known fact that an SBIBD(6, 3, 1) cannot exist).

We conclude that for each i, $r_i < 6$.

5. If $r_i = r_j = 5$, then $\lambda_{i,j} = 3$ — for if $\lambda_{i,j} = 5$ or 4 then $|A| = 0$ or ≤ 18 respectively, as in (3) and (6). Similarly, if $r_i = 5$ and $r_j = 4$ we cannot then have $\lambda_{i,j} = 3$ or 4 — else as above, $|A| \leq 27$ or 9 respectively. In any case, if $r_i = 5$ then every other row must have 1's in the columns corresponding to the 0's in $\mathbf{r}_i$. But then $|A| = 0$ — a contradiction.

   We conclude that $r_i = 4$, all i.
6. Together, steps 3 and 5 tell us that $AA^t = 2I + 2J$.

   We now can derive the following using the Rank-one Lemma:

$$\begin{aligned} |A| &= |AA^t|^{1/2} \\ &= |2I + 2J|^{1/2} \\ &= 2^{7/2} 8^{1/2} = 32. \end{aligned} \tag{7}$$

We have shown that if $A \in \mathcal{T}_7$ then either $|A| \leq 27$ or $|A| = 32$. The result follows. ∎

The existence of a $D_{32}(7)$ may be inferred from the existence of a Hadamard matrix of order 8. This provides the claimed counterexample.

3. Proliferation of (0, 1)-determinants.

Here are some of the best bounds on β_n:

$n \equiv 0 \pmod 4$: $\beta_n \leq (2n+1)^{1/2}(n/4)^{n/2}$ (Barba, [1]).

$n \equiv 1 \pmod 4$: $\beta_n \leq n((n-1)/4)^{\frac{n-1}{2}}$ (Ehlich [6], and Wojtas, [12]). Equality holds for $n \leq 100$ except $n = 21, 33, 57, 69, 73, 77, 89, 93, 97$ ([13], [3]).

$n \equiv 2 \pmod 4$: $\beta_n \leq (2n+1)^{1/2}(n/4)^{n/2}$ (Barba, [1]).

$n \equiv 3 \pmod 4$: $\beta_n \leq 2^{-n}(n+1)^{\frac{n+1}{2}}$ (this follows from the Hadamard bound and (1)). This bound is achieved precisely when there exists a Hadamard matrix of order $n + 1$ (this is established for many orders $\equiv 0 \pmod 4$ including all less than 428 [9]).

- There is also the lower bound $\beta_n > 2^{-n}(3(n+1)/4)^{\frac{n+1}{2}}$ for all n (Clements and Lindstrom, [4]; Schmidt [10] gives a sharper, but less straightforward, lower bound).

- We have $\beta_n = 1,1,2,3,5,9,32,56,144,320,1458,3645,9531$ for $n = 1,2,\ldots,13$ ([2], [11], [7]). $n = 14$ is apparently the first case for which β_n is not known.

The following main result for this section is given as one "monster lemma" since the different parts are designed to be used together recursively.

Lemma 3.1 (Monster Determinant Lemma, or MDL).

1. *$D_k(n)$ iff $D_{-k}(n)$ for all $n > 1$.*
2. *For any positive integers a, k and n, $D_k(n)$ implies $D_{ak}(n+a)$.*
3. *If $D_k(n)$ and $D_l(m)$ then*
   (i) $D_{kl}(m+n)$
   (ii) $D_{k^m l^n}(mn)$
   (iii) $D_{2^{mn} l^{n+1} k^{m+1}}(mn+m+n)$
   (iv) if the former is r-row-regular[5] and the latter has no row sum greater than r (for example, if it is s-row-regular with $s \leq r$ or if $m \leq r+1$), then there is an r-row-regular $D_{kl}(m+n)$.
4. *If there is an r-row-regular $D_k(n)$ and an s-row-regular $D_l(m)$ then*
   (i) there is an rs-row-regular $D_{k^m l^n}(mn)$
   (ii) there is a $(rm + sn - 2rs)$-row-regular $D_{2^{(m-1)(n-1)} k^m l^n (\frac{n}{r}-2)^{m-1} (\frac{m}{s}-2)^{n-1} (\frac{n}{r}+\frac{m}{s}-2)}(mn)$
   (iii) for $0 \leq d \leq n$, $0 \leq c \leq m$ there is a $D_{\frac{kl}{rs}(rs-cd)}(m+n)$ and a $D_{\frac{kl}{rs}(n(c-s)+m(d-r)+2rs-2cd)}(m+n-1)$
   (iv) for $-\min(r,s) \leq t \leq \min(n-r, m-s)$, there is an $(r+s+t)$-row-regular $D_{\frac{kl}{rs}t(r+s+t)}(m+n)$.
5. *If there is an r_i-row-regular $D_{k_i}(n_i)$ for each $i = 1,\ldots,s$ then*
   (i) $D_{k_1 \cdots k_s(\frac{a_1}{r_1}+\cdots+\frac{a_s}{r_s}-1)}(n_1+\ldots+n_s)$ for a_i satisfying $0 \leq a_i \leq n_i$
   (ii) $D_{k_1 \ldots k_s(\frac{n_1}{r_1}+\ldots+\frac{n_s}{r_s}-2)}(n_1+\ldots+n_s-1)$.
6. *There is av 1-regular $D_1(1)$.*
7. *If there is an r-row-regular $D_k(n)$ then there is an $(n-r)$-row-regular $D_{k(\frac{n}{r}-1)}(n)$.*
8. *If there is a Hadamard matrix of order $n+1$ (for example, any $n \equiv 3 \pmod 4$ and < 428), then there is an $(n+1)/2$-regular $D_{2(\frac{n+1}{4})^{\frac{n+1}{2}}}(n)$.*
9. *If there is a Hadamard matrix of order n and excess[6] x then $D_{(n/4)^{\frac{n}{2}}(1+x/n)}(n)$.*

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[5] a matrix R is *r-row(respectively column)-regular* if $RJ = rJ$ (respectively $JR = rJ$). A matrix is *r-regular* if it is both r-row and r-column-regular. In the lemma we shall disallow 0-row-regularity whenever this would imply division by 0.

[6] The *excess* of a matrix is the sum of its entries. Many such cases may be inferred from results in [5] — for example, all $x \equiv 0 \pmod 4$ satisfying $|x| \leq \max(n, 3n-8)$, and all $x = 2a\sqrt{n}$ for $|a| \leq n/2$ when n is the order of a regular Hadamard matrix, as is the case for each n where $\sqrt{n}$ is the order of a Hadamard matrix.

Proof:

1. This follows from the fact that interchanging two rows or columns negate a determinant (henceforth, this fact will be used without reference).
2. Suppose A is a $D_k(n)$. Expanding the following array by minors will verify that it is a $D_{\pm ak}(n+a)$:

$$\left(\begin{array}{c|ccc|cccc} 0 & 1 & \cdots & 1 & 1 & 0 & \cdots & 0 \\ \hline 1 & & & & & & & \\ \vdots & & I_{a-1} & & & & 0 & \\ 1 & & & & & & & \\ \hline v & & 0 & & & & A & \end{array}\right). \tag{8}$$

   Here v is the first column of A.
3. Let A be a $D_k(n)$ and B be a $D_l(m)$. Then
   (i) $|A \oplus B| = \begin{vmatrix} A & 0 \\ 0 & B \end{vmatrix} = |A|\,|B|$.
   (ii) $|A \otimes B| = |A|^m |B|^n$.
   (iii) using (1), construct (± 1)-matrices A' of order $n+1$ and B' of order $m+1$. Then verify that $A' \otimes B'$ is a (± 1)-matrix of order $mn+m+n+1$ with determinant $(2^n k)^{m+1}(2^m l)^{n+1}$. Then (1) implies the existence of the required matrix in $\mathcal{T}_{mn+m+n}$.
   (iv) If the row sums of B are all less than or equal to r, then we can choose C in $\mathcal{T}_n$ such that $B + C$ is r-row-regular, so then $\begin{pmatrix} A & 0 \\ C & B \end{pmatrix}$ an r-row-regular $D_{kl}(m+n)$.
4. Now further suppose that A is r-row-regular and B is s-row-regular. then
   (i) consider $A \otimes B$.
   (ii) apply the Rank-one lemma three times to $\frac{J_{mn} - (2A - J_n) \otimes (2B - J_m)}{2}$.
   (iii) choose C to be c-row-regular and D to be rank one and d-row-regular. Let $X = \begin{pmatrix} A & C \\ D & B \end{pmatrix}$. Using the Rank-one lemma (with $X = Y + \begin{pmatrix} 0 & 0 \\ D & 0 \end{pmatrix}$), we get $|X| = \frac{kl}{rs}(rs - cd)$, which is the first determinant. The reader may verify that $X^{-1} = Y^{-1} + \frac{rs}{cd - rs} Y^{-1} \begin{pmatrix} 0 & 0 \\ D & 0 \end{pmatrix} Y^{-1}$. Thus, $tr(X^{-1} J_{m+n}) = \frac{n(c-b) + m(d-a)}{cd - rs}$. From this we may use the Rank-one lemma again to show that $2X - J_{m+n}$ is a (± 1)-matrix with determinant $2^{m+n} \frac{kl}{rs}\left(\frac{n(c-s) + m(d-r)}{2} + rs - cd\right)$, which together with (1) establishes the second determinant.

(iv) Now if $-\min(r,s) \le t \le \min(n-r, m-s)$, we can find c and d as above with $r+c = s+d = r+s+t$. Then X is $r+s+t$-row-regular and has determinant $-\frac{kl}{rs}t(r+s+t)$.

5. Now let $X_1, \dots, X_s$ be, respectively, r_i-row-regular $D_{k_i}(n_i)$'s for $i = 1, \dots, s$. For compactness, write $J^a_{m,n}$ for the $m \times n$ matrix whose entries in the first a columns are all 1 and the rest of whose entries are 0. Then take

$$M = \begin{pmatrix} X_1 & & 0 \\ & \ddots & \\ 0 & & X_s \end{pmatrix} \text{ and } U = \begin{pmatrix} J^{a_1}_{n_1 \times n_1} & \cdots & J^{a_s}_{n_1 \times n_s} \\ \vdots & & \vdots \\ J^{a_1}_{n_s \times n_1} & \cdots & J^{a_s}_{n_s \times n_s} \end{pmatrix}. \tag{9}$$

(i) It is easy to verify that

$$M^{-1}U = \begin{pmatrix} \frac{1}{r_1} J^{a_1}_{n_1 \times n_1} & \cdots & \frac{1}{r_1} J^{a_s}_{n_1 \times n_s} \\ \vdots & & \vdots \\ \frac{1}{r_s} J^{a_1}_{n_s \times n_1} & \cdots & \frac{1}{r_s} J^{a_s}_{n_s \times n_s} \end{pmatrix}. \tag{10}$$

Thus $tr(M^{-1}U) = \frac{a_1}{r_1} + \cdots + \frac{a_s}{r_s}$, and so $|M-U| = |M||I - M^{-1}U| = k_1 \dots k_s(1 - (\frac{a_1}{r_1} + \cdots + \frac{a_s}{r_s}))$, by an application of the Rank-one lemma. Negating the appropriate columns gives a matrix in $\mathcal{T}_n$ whose determinant has the same norm.

(ii) When $a_i = n_i$, all i, then $U = J$, so $tr(M^{-1}J) = \frac{n_1}{r_1} + \cdots + \frac{n_s}{r_s}$. Apply (1) to $2M - J$.

6. This is clear.
7. Now let A be an r-row-regular $D_k(n)$. Then $J - A$ is $(n-r)$-row-regular and the Rank-one lemma gives $|J - A| = \pm|A|\ |I - A^{-1}J| = \pm k(1 - tr(\frac{1}{r}J)) = \pm k(\frac{n}{r} - 1)$.
8. This follows directly from the construction in (1), and the fact that the given Hadamard matrix has determinant $(n+1)^{\frac{n+1}{2}}$.
9. Apply the Rank-one lemma to $\frac{J+H}{2}$, where H is the given Hadamard matrix.

∎

The following two theorems are just a simple demonstration of the power of this lemma:

Theorem 3.1. *For each n and $0 \le k < n$, there is a k-row-regular $D_k(n)$.*

Proof: I_{k+1} is a 1-regular $D_1(k+1)$. A k-regular $D_k(k+1)$ follows by part 7 of MDL. The result follows by using this matrix together with I_{n-k-1} in part 3iv). ∎

Theorem 3.2. *If there is an r-row-regular $D_k(n)$ then $r|k$.*

Proof: Using this matrix and I_1 in part 3iv), we get an r-row-regular $d_k(n+1)$. Thus there is a $D_{k(\frac{n}{r}-1)}(n)$ and a $D_{k(\frac{n+1}{r}-1)}(n+1)$, by part 7. So $r|kn$ and $r|k(n+1)$ and so $r|k$. ∎

Table 1 compiles the results given by MDL for $n \leq 10$, comparing them to determinants found using other methods ("–" indicates an entire range of determinants which are established, the left endpoint defaulting to 0).

Table 1

Determinants found up to order 10 with and without MDL

n	MDL implies $D_k(n)$ for $k =$	other determinants found without MDL
1	−1	
2	−1	
3	−2	
4	−3	
5	−5	
6	−8	9
7	−13, 16, 18, 20, 24, 32	14, 15, 17, 19
8	−18, 20, 24, 28, 32, 40, 48, 56	19, 21 − 23, 25 − 27, 29 − 31, 33 − 39, 42, 44, 45
9	−24, 26, 28, 32, 40, 48, 56, 64, 72, 80, 88, 144	25, 27, 29 − 31, 33 − 39, 41 − 47, 49 − 55, 57 − 63, 65 − 71, 73 − 79, 89 − 102, 104, 105, 108, 110, 112, 116, 120, 125, 128
10	−34, 36, 38 − 40, 44, 48, 56, 64, 72, 80, 88, 96, 104, 112, 120, 128, 136, 144	35, 37, 41 − 43, 45 − 47, 49 − 55, 57 − 63, 65 − 71, 73 − 79, 81 − 87, 89 − 95, 97 − 103, 105 − 111, 113 − 119, 121 − 127, 129 − 135, 137 − 143, 145 − 232, 234 − 256, 258, 260, 261, 263 − 267, 270, 272 − 276, 279, 280, 283 − 285, 288, 291, 294 − 297, 304, 312, 315, 320

4. What next?

With the demise of the consecutive integer determinant conjecture, several related questions are left wide open:

- What is the range of the determinant function on $\mathcal{T}_n$ in general, and the value of β_n in particular?
- It appears from constructions so far that very strong structure is dictated for matrices of large determinant in $\mathcal{T}_n$. For example, if a Hadamard matrix of order $n+1$ exists, a $D_{\beta_n}(n)$ must be an SBIBD(n, $\frac{n+1}{2}$, $\frac{n+1}{4}$) (or rather, the $(0,1)$-incidence matrix corresponding to it — however, we shall not distinguish between them here). Now consider a matrix from $\mathcal{T}_n$ of the form $\left(\begin{array}{c|c} 1 \cdots 1 & 0 \cdots 0 \\ \hline A & B \end{array}\right)$. We can show that it has, up to sign, the same determinant as $\left(\begin{array}{c|c} 1 \cdots 1 & 0 \cdots 0 \\ \hline J-A & B \end{array}\right)$. If we combine transformations of this type with row and column permutation and the transpose operation, these together define an equivalence relation in $\mathcal{T}_n$ with respect to which the absolute value of the determinant is invariant. With this notion of equivalence it is possible to show that s $D_1(2)$ is equivalent to I_2, a $D_2(3)$ is equivalent to $J_3 - I_3$ (that is, an SBIBD(3, 2, 1)), a $D_3(4)$ is equivalent to an

SBIBD(4, 3, 2), or alternately,

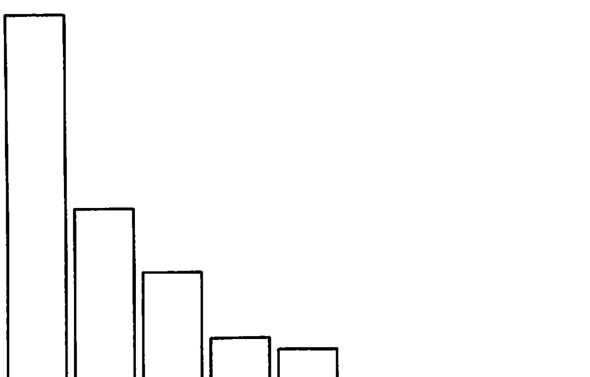
$$\left(\begin{array}{c|ccc} 0 & 1 & 1 & 1 \\ \hline 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right).$$

Similarly, $D_5(5)$ is equivalent to

$$\left(\begin{array}{cc|ccc} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ \hline 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{array}\right).$$

Are there principles which dictate, up to equivalence, the structure of a $D_{\beta_n}(n)$, or in general any "large enough" determinant in any given order?

- With what frequency does each determinant occur in each order? Figure 1 compiles results obtained from a fairly large sample in order 7.



Determinant	0	1	2	3	4	5	6	7	8	9	10
Frequency	529	239	144	46	28	4	4	3	1	1	1

Figure 1
Occurrences of determinants of order 7 among a sample of 1000

- With regard to the relationship between row-regularity and determinants, there are a number of questions. Chief among these is the following: for what triples (r, n, k) do there exist an r-row-regular $D_k(n)$? Some elementary partial results:
  - (r, n, k) must satisfy $0 \leq r \leq n$, $k \leq \beta_n$, $r|k$. If $r = 0$ or n, $k = 0$; if $r = 1$, $k = 1$; if $r = n-1$, $k = n-1$.

- if $r|k$ and $D_p(r+1)$ for every factor p in some factorization of k then there is an n for which (r, n, k) is such a triple (this is almost certainly true when the second condition is replaced with $r > 1$). In particular, when $r \geq \sqrt{k}$ we may take $n \leq 2k$.
- if (r, n, k) is such a triple and $n' > n$ then (r, n', k) is one as well.

• For each n, let β'_n be the largest number such that $k \leq \beta'_n$ implies $D_k(n)$. In other words, $[-\beta'_n, \beta'_n]$ is the largest possible interval of consecutive integer $(0,1)$-determinants in order n centered at 0. We have shown that $\beta'_7 \neq \beta_7$. Can we have $\beta'_n = \beta_n$ for $n > 7$? What are some non-trivial bounds on β'_n? The only *upper* bound I know so far is β_n. A trivial lower bound from Theorem 3.1 is $\beta'_n \geq n-1$. Here is a somewhat better one:

Theorem 4.1. *For all n, $\beta'_n \geq \lfloor \frac{n}{2} \rfloor^2 - 1$.*

Proof: Write $u = \lfloor \frac{n}{2} \rfloor$. If $m \leq u$ then Theorem 3.1 provides a $D_m(n)$. If $u < m \leq u^2 - 1$, we can write $m = s(u-1)+t$, where $1 \leq s \leq u$ and $1 \leq t \leq u$. Now take an s-row-regular $D_s(u)$ and I_{n-u} and apply part 5i) of MDL, with $a_1 = t$ and $a_2 = u$. This gives us a $D_{1 \cdot s(\frac{t}{s}+\frac{u}{1}-1)}(u + (n-u)) = D_m(n)$. ∎

This theorem actually shows that part 2 of MDL is not neccesary for $a > 4$, since then $\beta'_a > a$, and so the first point of part 3 gives the same result (we note that part 2 is not used to establish this result).

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