Neighborhood and Self-Dual (Multi)Graphs

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Abstract. We introduce neighborhood intersection graphs and multigraphs of loop-graphs to generalize the standard notions of square and distance-two graphs. These neighborhood (multi)graphs are then used to construct self-dual graphs and multigraphs (embedded on surfaces of varying genus) which have involutory vertex-face mappings.

1. Neighborhood multigraphs of loop-graphs.

A *loop-graph* is a finite graph with loops possible at vertices. (See [5] for this and any unexplained terminology. For convenience, we assume all graphs, loop-graphs and multigraphs are connected.) If L is any loop-graph with vertex set $\{v_1, \dots, v_n\}$, the *neighborhood* N_i of vertex v_i is the set of all vertices adjacent to v_i, including v_i itself if and only if there is a loop at v_i. The *neighborhood multigraph* $N(L)$ of L is the intersection multigraph of the family of neighborhoods $N_1, \dots, N_n$. We think of $N(L)$ as having the same vertex set as L and give an example in Figure 1.

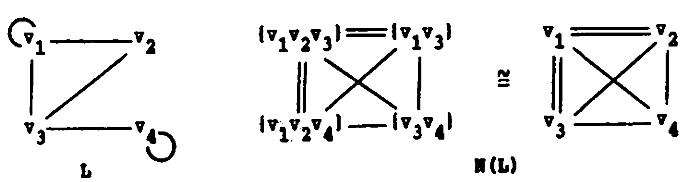

Figure 1

The *neighborhood graph* of L is the intersection graph of its neighborhoods, and so is the underlying graph of the neighborhood multigraph; we denote it by $N(L)\downarrow$. If G is any graph and L is G with a loop added at each vertex, then $N(L)\downarrow$ is the *square* (or *closed-neighborhood graph*) of G. If G is any graph and L is identical to G (i.e., if L is loopless), then $N(L)\downarrow$ is the *distance-two graph* (or *two-step graph* or *open-neighborhood graph*) of G. Our notion subsumes both these as extreme cases, as well as extending them to the stronger domain of intersection multigraphs. (Related extensions of various kinds of intersection

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graphs to intersection multigraphs are found in [1, 8, 9, 10, 11, 12]. There are advantages in both theory and applications to using multigraphs rather than just their underlying graphs.) In the second section of this paper, we show an application of neighborhood multigraphs to finding self-dual graphs embedded on surfaces; this is extended to multigraphs in the third section.

Neighborhood graphs and distance-two graphs were characterized by Mukhopadhyvy [14] and Acharya and Vartak [4], respectively. These characterizations are generalized to neighborhood multigraphs of loop-graphs in Theorem 1 below.

Theorem 1. *A multigraph M of order n is the neighborhood multigraph of a loop-graph if and only if its edgeset can be partitioned by a collection $C_1, \ldots, C_n$ (allowing repetitions) of complete (simple) subgraphs such that, for each $1 \leq i, j \leq n$, $v_i \in V(C_j)$ if and only if $v_j \in V(C_i)$.*

Proof: We merely sketch the proof since it is very similar to the characterizations in [14] and [4]. Take each $V(C_i)$ to be N_i. The key observation (in the spirit of [8]) is that for any $1 \leq i, j \leq n$, $v_i \in N_j$ if and only if $v_j \in N_i$. ∎

To obtain the characterization of neighborhood graphs from Theorem 1, simply replace "partitioned" with "covered". To obtain the characterization of closed [or open] neighborhood graphs, add the requirement that $v_i \in V(C_i)$ [respectively, $v_i \notin V(C_i)$], for each $1 \leq i \leq n$.

Paralleling the extension of other types of intersection graphs to multigraphs, it is natural to seek a simple graph which is a neighborhood graph, but which is not a neighborhood multigraph when viewed as a multigraph with each multiplicity equal to one. The graph G shown in Figure 2 is such an example. This is the neighborhood graph of the loop-graph L shown here in Figure 2.

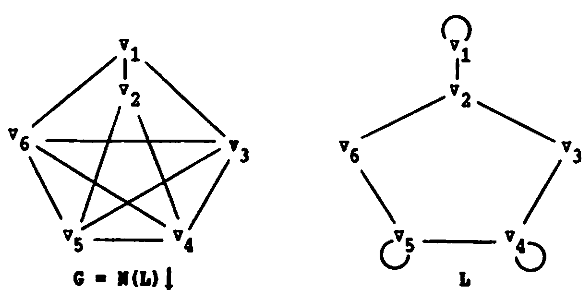

Figure 2

(The only difference between $N(L)$ and G is that $N(L)$ would have a double edge between v_4 and v_5.) But Theorem 1 shows that G is not a neighborhood multigraph.

Proof: Suppose, rather, there were a collection $C_1, \ldots, C_6$, as in Theorem 1. None of the C_i could be K_4, lest the other five members of the collection be K_2's and each vertex be in either two or three members; some vertex needs to be in four to match with the K_4. So the collection would have to contain two edge-disjoint K_3's and five K_2's, or one K_3 and eight K_2's, but either requires more than six C_i's. ∎

Because of the increased flexibility allowed by using loop-graphs, it is hoped that a "nicer" characterization might exist for neighborhood graphs or multigraphs than in the traditional setting with all open or all closed neighborhoods. By "nicer" we mean that it would not involve the existence of a particular family of subgraphs. (In other words, it would be farther from the definition, in the sense of not being predictable using the mechanism of [8].) While we have no such nicer characterization to offer, the next section does give an application of neighborhood multigraphs for which the context of loop-graphs is ideal.

2. Application to self-dual graphs.

Suppose M is any multigraph with vertex set $\{v_1, \ldots, v_n\}$ which has been *embedded* on a surface Σ. (We assume all our embeddings are 2-cell embeddings, as in [5, 16], and that, whatever the genus, they are "polyhedral maps" in that each edge is on two faces.) Suppose that the faces of this embedding are indexed as $F_1, \ldots, F_n$. We say that the pairing of each vertex v_i with face F_i is an *involutory vertex-face mapping* for M on Σ whenever, for any $1 \leq i, j \leq n$, $v_i \in V(F_j)$ if and only if $v_j \in V(F_i)$. We say that the matching of each v_i with F_i is a *self-dual mapping* whenever the number of (parallel) edges common to vertices v_i and v_j always equals the number of edges common to faces F_i and F_j. We call an embedded (multi)graph *self-dual* whenever it admits a self-dual mapping.

In this section we consider only the case that M is a simple graph G, postponing the general multigraph case to the next section. Whenever G is 3-connected, each involutory vertex-face mapping is automatically a self-dual mapping. But [7] gives an example of a 3-connected graph which has a self-dual mapping, yet no involutory self-dual mapping. (The existence of such a graph was unknown as recently as [6]; see [2, 3] for updated discussions.)

Theorem 2. *Suppose G is any 3-connected graph having an involutory vertex-face mapping on a surface Σ. Then we can construct a loop-graph L such that $G \cong N(L) - N(L)\downarrow$ (i.e., G is $N(L)$ with each multiplicity decreased by one).*

Proof: Suppose G is embedded on the surface Σ with each v_i paired with F_i by an involutory mapping. Construct a loop-graph L having vertex set $\{v_1, \ldots, v_n\}$ with an edge between v_i and v_j (allowing i to equal j) if and only if v_i is incident with F_j in G.

Define a bipartite graph R (called the *radial graph*, introduced in [15, Section 3.5] for the case of a plane graph G) to have vertex set $\{v_1, \ldots, v_n; F_1, \ldots, F_n\}$, with an edge between vertices v_i and F_j if and only if $v_i \in V(F_j)$. The bipartite graph R can be thought of as combining both the embedding of G and the involutory vertex-face mapping. (Note that both G and its surface dual on Σ determine the same radial graph.)

Toward showing that $G \cong N(L) - N(L) \downarrow$, suppose v_i is adjacent with v_j in G. Two faces F_p and F_q in G incident with edge v_iv_j must be the only faces incident with both v_i and v_j (since G is 3-connected). Hence in R, v_i and v_j will be joined by exactly two paths of length two (through F_p and F_q), so they will be joined by a double edge in $N(L)$ and so by a single edge in $N(L) - N(L) \downarrow$. If v_i is not adjacent to v_j in G, then (again using 3-connectedness) they will be on at most one common face of G, so joined by at most one path of length two in R, so joined by at most a single edge in $N(L)$, and so nonadjacent in $N(L) - N(L) \downarrow$. ∎

As an example, consider the involutory mapping indicated by the placement of the F_i's in the wheel graph G in Figure 3. The neighborhoods of L (e.g., $N_1 = \{v_1, v_2, v_6\}$) correspond exactly with the faces in the plane embedding of $N(L) - N(L) \downarrow$. (A different involutory mapping for G would take each F_i to be the face "directly across" from v_i; the resulting L would be a loopless wheel, again with $G \cong N(L) - N(L) \downarrow$.) Another example, on the torus, is shown in Figure 4. An interesting consequence of [2, Theorem 3.7] is that whenever G is a 3-connected, planar, involutorily self-dual graph, then L must have an even number of loops.

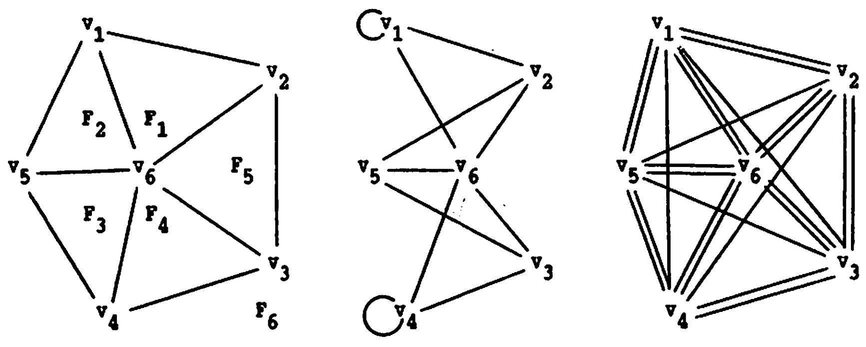

Figure 3

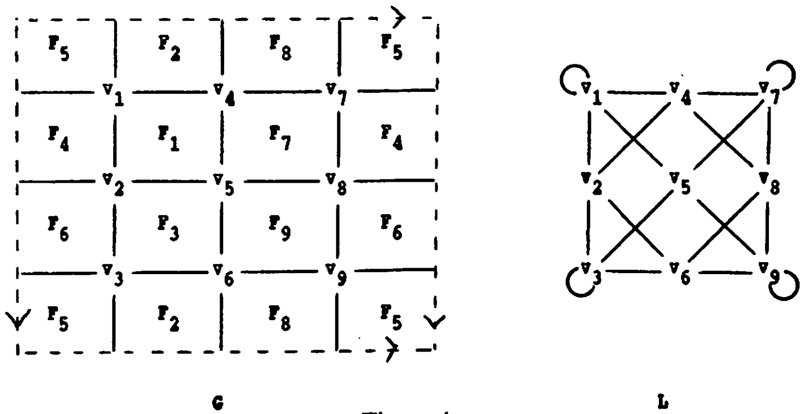

Figure 4

In practice, it is useful to use Theorem 2 backwards: A bounty of (always involutorily) self-dual graphs can be generated by starting with an arbitrary loop-graph L, then using the neighborhoods of L to determine the faces of an embedding of $N(L) - N(L) \downarrow$ in a surface. Indeed, that is how the above examples were generated.

3. Application to self-dual multigraphs.

Although Theorem 2 was stated only for simple graphs, it holds as well for multigraphs of order ≥ 3, since the only such 3-connected multigraph which has such an involutory vertex-face mapping is the M shown in Figure 5. In this section, we consider the case of multigraphs in general.

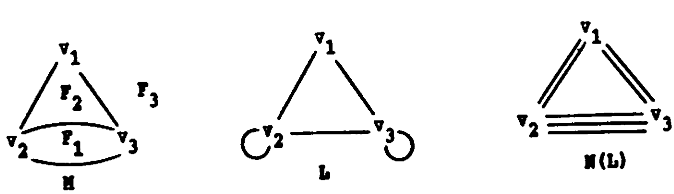

Figure 5

Suppose L is any loop-graph with vertex set $\{v_1, \ldots, v_n\}$. Suppose M is any multigraph, on the same vertex set, which has been embedded on a surface Σ such that the neighborhoods $N_1, \ldots, N_n$ of L correspond precisely to the vertex sets of (all) the faces $F_1, \ldots, F_n$ of the embedding of M on the surface. Notice that, when $1 \leq i, j \leq n$, $v_i \in V(F_j)$ is equivalent to $v_i \in N_j$, and so to $v_j \in N_i$, and so to $v_j \in V(F_i)$. Thus pairing each v_i with F_i defines an involutory vertex-face mapping of M on Σ.

Given such a multigraph embedded on Σ, it is easy to produce a loop-graph L (from the bipartite radial graph R) as in the proof of Theorem 2. Theorem 3 shows how M can be recovered from L using its neighborhood multigraph, generalizing the formation of $N(L) - N(L)\downarrow$ from Theorem 2. For any vertices v_i and v_j of M, let $\mu(v_i, v_j)$ be the number of parallel edges of M which contain the two vertices, and $\rho(v_i, v_j)$ be the number of faces of the embedding which contain both.

Theorem 3. *Suppose M is any multigraph having an involutory vertex-face mapping on a surface* Σ. *Then we can construct a loop-graph L such that M can be obtained from* $N(L)$ *by deleting precisely* $\rho(v_i, v_j) - \mu(v_i, v_j)$ *parallel edges from those joining* v_i *with* v_j, *for each pair of adjacent vertices in* $N(L)$.

Proof: Suppose M is a multigraph embedded in Σ with each v_i matched with F_i by an involutory mapping. Define the bipartite graph R and loop-graph L as in the proof of Theorem 2. Suppose v_i is connected with v_j in M by μ (possibly zero) parallel edges, and that they are in ρ (necessarily $\geq \mu$) common faces. Hence in R, v_i and v_j will be connected by precisely ρ paths of length two, so by ρ edges in $N(L)$. Deleting $\rho - \mu$ edges will leave v_i connected with v_j by μ edges in the multigraph. ∎

The following examples show the relation between M and L. Figure 6 shows a self-dual plane multigraph M with the indicated involutory vertex-face mapping. Note that v_2 and v_6 are joined by three parallel edges in $N(L)$, so $\rho(v_2, v_6) - \mu(v_2, v_6) = 3 - 2 = 1$ of them are deleted in recovering M.

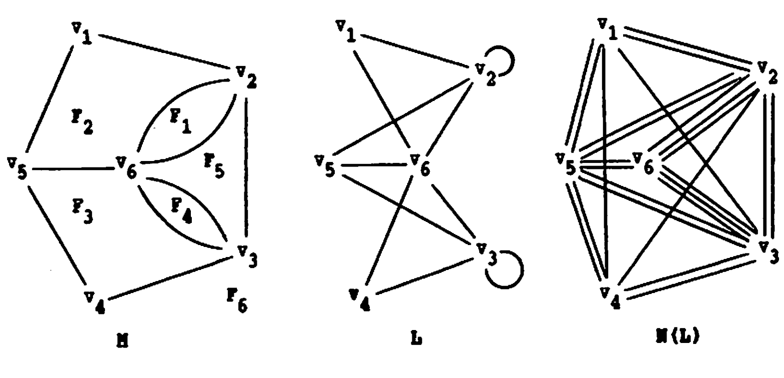

Figure 6

Corollary. *Suppose M is any multigraph having an involutory vertex-face mapping on a surface* Σ. *Then we can construct a loop-graph L such that M is a spanning submultigraph of* $N(L)$.

Once again, it is useful in practice to use Theorem 3 backwards, generating multigraphs having involutory vertex-face mappings (and so potential self-dual multigraphs) by starting with an arbitrary loop-graph L, identifying the submultigraphs of $N(L)$ which correspond to the neighborhood of L, and then trying to delete edges until these submultigraphs can be embedded as the faces on some surface; the example in Figure 6 was obtained in this fashion. An additional example starts with the loop-graph L in Figure 7. The neigborhood multigraph $N(L)$ indicates that the desired embedding would have to have five faces having vertex sets $\{v_2, v_3, v_4, v_5\}$, $\{v_1, v_2, v_3, v_4, v_5\}$, $\{v_1, v_2\}$, $\{v_1, v_2\}$ and $\{v_1, v_2\}$. Edge deletion (and repositioning the vertices on the plane) produces the multigraph M. (Every multigraph with an involutory vertex-face mapping given so far as an example in this paper is self-dual. This need not happen in general, but the author is lacking such an example on the plane.)

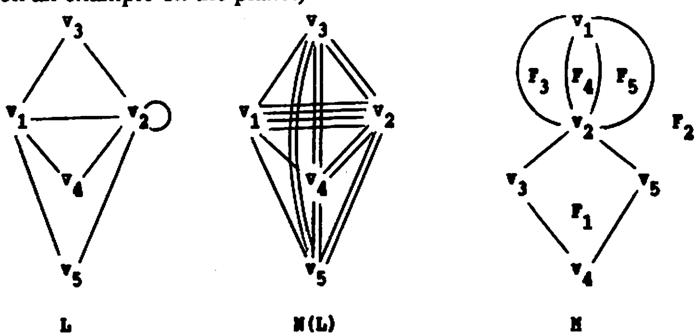

Figure 7

Of course, given an arbitrary loop-graph L the process in the preceding paragraph (or in the final paragraph of Section 2) may well fail; a (loopless) example is shown in Figure 8. Trying to follow the pattern in Theorem 3 yields the hypergraph H shown in Figure 8, in which $\{v_1, v_2, v_3,\}$ and $\{v_4, v_5, v_6\}$, are hyperedges. This is, in some sense, a self-dual embedding of the hypergraph, and in general self-dual embeddings of self-dual "multi-hypergraphs" are produced.

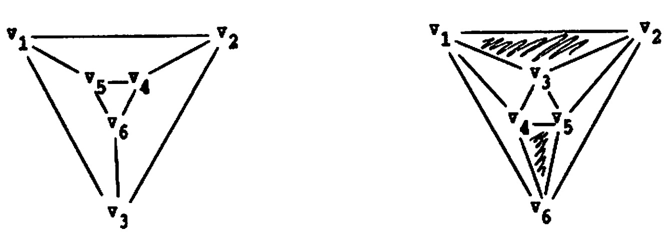

Figure 8

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