Perfect Mendelsohn Designs With Equal-Sized Holes

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Abstract. It is shown that the obvious necessary condition $\lambda h(h-1)m^2 \equiv 0 \pmod{k}$ for the existence of a (v,k,λ)-perfect Mendelsohn design with h holes of size m is sufficient in the case of block size three except for a nonexisting (6,3,1)-PMD.

1. Introduction

By $\lambda K_{n_1,n_2,\ldots,n_h}$ we mean a complete multipartite directed multigraph on a vertex set $X = \cup_{1\le i\le h} X_i$, where $X_i (1 \le i \le h)$ are disjoint sets with $|X_i| = n_i$, $v = \sum_{1\le i\le h} n_i$ and where two vertices x and y from different sets X_i and X_j are joined by two arcs (x,y) and (y,x) exactly λ times each.

If $\lambda K_{n_1,n_2,\ldots,n_h}$ can be decomposed into directed k-circuits such that for any r, $1 \le r \le k-1$, and for any two vertices x and y from different sets X_i and X_j, there are exactly λ circuits along which the (directed) distance from x to y is r, we call $(X, \mathbf{B})$ a *holey perfect Mendelsohn design*, where $\mathbf{B}$ is the collection of all circuits (called blocks). We denote the design by (v,k,λ)-HPMD. Each set $X_i (1 \le i \le h)$ is called a *hole* and the vector $(n_1, n_2, \ldots, n_h)$ is called the *type* of the HPMD.

A (v,k,λ)-HPMD of type $(1,1,\ldots,1)$ is called a *perfect Mendelsohn design* denoted by (v,k,λ)-PMD. If we ignore the cyclic order of the vertices in the circuits, a (v,k,λ)-PMD becomes a $(v,k,\lambda(k-1))$-BIBD. Therefore, we can consider a perfect Mendelsohn design as a generalization of balanced incomplete block designs. It was N.S. Mendelsohn who first introduced the cyclic order of the elements into blocks (see [4], [5]). The existence question of a (v,k,λ)-PMD has recently attracted much attention, and a survey can be found in [6]. The concept of HPMD has played an important role in the discussion of the existence of PMDs. Further, the existence question for a (v,k,λ)-HPMD is also posed in [6]. In this paper we consider only the simple case of equal-sized holes, i.e., $n_1 = n_2 = \cdots = n_h = m$. The type will be denoted (m^h). Since a (v,k,λ)-HPMD of type

(m^h) contains $b = \lambda h(h-1)m^2/k$ blocks, we obtain the following necessary condition:

$$\lambda h(h-1)m^2 \equiv 0 \pmod{k}. \tag{1.1}$$

We shall show that (1.1) is also sufficient for the existence of a (v,k,λ)-HPMD of type (m^h) when $k=3$ except for a nonexisting (6,3,1)-PMD.

2. Preliminaries

We assume that the reader is familiar with the group divisible designs (GDDs), pairwise balanced designs (PBDs), transversal designs (TDs), etc. For more information the reader is referred to [3]. We shall often use the following weighting techniques which can be found in [6] for the case of $\lambda = 1$. Here we omit their proofs, and further details can be found in [2].

Lemma 2.1. *Suppose there exists a* TD$[k;m]$, *then*

(1) *there exists an* (mn,k,λ)*-HPMD of type* $(mn_1, mn_2, \ldots, mn_h)$ *if there exists an* (n,k,λ)*-HPMD of type* $(n_1, n_2, \ldots, n_h)$*; and*
(2) *there exists an* (mn,k,λ)*-HPMD of type* (m^n) *if there exists an* (n,k,λ)*-PMD.*

Lemma 2.2. *Suppose there is a* GDD$[K,1,M;v]$ *with groups* $G_1, G_2, \ldots, G_h$ *where* $M = \{|G_i| = n_i \mid 1 \le i \le h\}$*. If for any block size* $u \in K$ *there is an* (mu,k,λ)*-HPMD of type* (m^u)*, then there exists an* (mv,k,λ)*-HPMD of type* $(mn_1, mn_2, \ldots, mn_h)$.

The following result on GDDs (see [3, p. 466]) is useful.

Lemma 2.3. *A* GDD$[k,\lambda,g;gs]$ *exists for* $k=3$ *or* 4 *if and only if*

$$\begin{aligned} \lambda(s-1)g &\equiv 0 \pmod{k-1}, \\ \lambda s(s-1)g^2 &\equiv 0 \pmod{k(k-1)}, \end{aligned} \tag{2.1}$$

except for two nonexisting designs GDD$[4,1,2;8]$ *and* GDD$[4,1,6;24]$.

The existence of a $(v,3,\lambda)$-PMD has been established in [1],[4] and an alternative proof can be found in [6].

Lemma 2.4. *A necessary and sufficent condition for the existence of a* $(v,3,\lambda)$*-PMD is* $\lambda v(v-1) \equiv 0 \pmod 3$*, except for a nonexisting design* $(6,3,1)$*-PMD.*

We also need a result on RBIBDs (see [3, p.451]).

Lemma 2.5. *An RBIBD* RB$[3,1;v]$ *exists if and only if* $v \equiv 3 \pmod 6$.

3. Main results

When $k = 3$, (1.1) becomes $\lambda h(h-1)m^2 \equiv 0 \pmod 3$. This can be divided into two cases:

(i) $m \equiv 0 \pmod 3$ and $h \geq 3$;

(ii) $m \not\equiv 0 \pmod 3$ and $\lambda h(h-1) \equiv 0 \pmod 3$.

Proposition 3.1. *For $m \equiv 0 \pmod 3$ and $h \geq 3$, there is an $(mh,3,1)$-HPMD of type (m^h).*

Proof: Write $m = 3t$. Since a TD[3; t] exists for $t \geq 2$ by Lemma 2.3, by applying Lemma 2.1 (1) we reduce the problem to the case of $m = 3$. Let $H = \{h \geq 3 \mid h \text{ is an integer}\}$. It is known [3] that

$$H = B(\{3,4,5,6,8\}), \tag{3.1}$$

that is, for every $h \in H$ there exists a PBD $(X, \mathbf{B})$ where $|X| = h$ and $|B| \in \{3,4,5,6,8\}$ for every block $B \in \mathbf{B}$

From (3.1) we have a GDD[$\{3,4,5,6,8\}, 1, \{1\}; h$] for any $h \geq 3$. By Lemma 2.2 we need only to show the existence of a $(3s,3,1)$-HPMD of type (3^s) for $s \in \{3,4,5,6,8\}$. For $s = 3,4,5,8$, we apply Lemma 2.3 to obtain a GDD[$k,1,3;3s$] where $k = 3$ or 4. For $s = 6$, we start with an RBIBD RB[$3,1;15$] and consider it as an RGDD[$3,1,3;15$]. Taking three parallel classes and adding a new point to the blocks of each parallel class, we obtain a GDD[$\{3,4\}$, $1,3;18$]. For all these GDD[$\{3,4\},1,3;3s$], $s \in \{3,4,5,6,8\}$, we construct either a (3,3,1)-PMD or a (4,3,1)-PMD on each block and obtain a $(3s,3,1)$-HPMD of type (3^s). Here the required $(k,3,1)$-PMD for $k = 3,4$ comes from Lemma 2.4. The proof is now complete.

Corollary 3.2. *For $m \equiv 0 \pmod 3$ and $h \geq 3$, there is an $(mh,3,\lambda)$-HPMD of type (m^h) for any positive integer λ.*

Proof: The proof follows directly from Proposition 3.1 by taking repeated blocks.

Proposition 3.3. *For $m \not\equiv 0 \pmod 3$, $h \neq 6$ and $\lambda h(h-1) \equiv 0 \pmod 3$, there is an $(mh,3,\lambda)$-HPMD of type (m^h).*

Proof: By Lemma 2.4 an $(h,3,\lambda)$-PMD exists when $\lambda h(h-1) \equiv 0 \pmod 3$ and $h \neq 6$. This solves the case of $m = 1$. It is also known by Lemma 2.3 that a TD[3; m] exists for $m \geq 2$. Applying Lemma 2.1 (2), we obtain an $(mh,3,\lambda)$-HPMD of type (m^h). The proof is complete.

We shall now concentrate on the remaining case of $m \not\equiv 0 \pmod 3$, $h = 6$ and any λ.

Lemma 3.4. *A $(12t,3,\lambda)$-HPMD of type $((2t)^6)$ exists for any $t \geq 1$.*

Proof: Since a GDD[$3,\lambda,2t;12t$] exists from Lemma 2.3, we can construct a (3,3,1)-PMD on each block to obtain the required HPMD.

For $m \not\equiv 0 \pmod 3$ and $2 \nmid m$, we can write $m = 6k+5$ or $6k+7$, $k \geq 0$.

Lemma 3.5. *A* $(6m,3,1)$*-HPMD of type* (m^6) *exists for* $m = 6k+5 \geq 11$.

Proof: First, we have an RGDD$[3,1,3;6k+3]$ from Lemma 2.5. Adding new points to two of the parallel classes and breaking up the blocks of sizes 3 and 4 with a (3,3,1)-PMD and a (4,3,1)-PMD, we obtain a $(6k+5,3,1)$-HPMD of type $(3^{2k+1}2^1)$, say $(X,\mathbf{B})$ where X_0 is a hole of size two and $X_1,\ldots,X_{2k+1}$ are holes of size three. Let $I = \{1,2,\ldots,6\}$. We need the following idempotent Latin square of order 6:

$$A = (a_{ij}) = \begin{bmatrix} 1 & 3 & 2 & 5 & 6 & 4 \\ 4 & 2 & 5 & 6 & 1 & 3 \\ 6 & 5 & 3 & 1 & 4 & 2 \\ 2 & 6 & 1 & 4 & 3 & 5 \\ 3 & 4 & 6 & 2 & 5 & 1 \\ 5 & 1 & 4 & 3 & 2 & 6 \end{bmatrix}$$

Next, for each block $B = (a,b,c)$ in the HPMD we form a set of new blocks

$$\mathbf{D}_B = \{((a,i),(b,j),(c,a_{ij})) \mid i,j \in I\}.$$

Since A is idempotent, $\mathbf{D}_B$ contains a subset

$$I_B = \{((a,i)(b,i),(c,i)) \mid i \in I\}.$$

Let $\mathbf{B}_B = \mathbf{D}_B \backslash I_B$.

By Lemma 2.1(1), we know that $(X \times I, \cup_{B\in\mathbf{B}} \mathbf{D}_B)$ is a $(6m,3,1)$-HPMD of type $(18^{2k+1}12^1)$ with holes $X_j \times I, 0 \leq j \leq 2k+1$.

Finally, using Proposition 3.1 and Lemma 3.4, we construct an (18,3,1)-HPMD or a (12,3,1)-HPMD of type (3^6) or (2^6), say $(X_j \times I, \mathbf{A}_j)$ for $0 \leq j \leq 2k+1$ having holes $X_j \times \{i\}, i \in I$. Therefore, $(X \times I, (\cup_{B\in\mathbf{B}} \mathbf{D}_B) \cup (\cup_{0\leq j\leq 2k+1} \mathbf{A}_j))$ is a $(6m,3,1)$-HPMD of type $(3^{6(2k+1)}a2^6)$ having holes $X_j \times \{i\}, i \in I$ and $0 \leq j \leq 2k+1$. This HPMD contains six sub-HPMD $(X \times \{i\}, \{((a,i),(b,i),(c,i)) \mid (a,b,c) \in \mathbf{B}\})$ each isomorphic to the original $(X,\mathbf{B})$. Omitting the blocks in these sub-HPMDs, we know that $(X \times I, (\cup_{B\in\mathbf{B}} \mathbf{B}_B) \cup (\cup_{0\leq j\leq 2k+1} \mathbf{A}_j))$ is a $(6m,3,1)$-HPMD of type (m^6) having holes $X \times \{i\}, i \in I$. This completes the proof.

Corollary 3.6. *For* $m = 6k+5 \geq 11$, *there exists a* $(6m,3,\lambda)$*-HPMD of type* (m^6) *for every positive integer* λ.

Proof: The proof follows directly from Lemma 3.5 by taking repeated blocks.

Lemma 3.7. *A* $(6m,3,\lambda)$*-HPMD of type* (m^6) *exists for* $m = 6k+7 \geq 13$.

Proof: The proof is similar to that of Lemma 3.5 and Corollary 3.6. In this case, the starting HPMD becomes a $(6k+7,3,1)$-HPMD of type $(3^{2k+1}4^1)$. The required (24,3,1)-HPMD of type (4^6) comes from Lemma 3.4.

Lemma 3.8. *A* $(30,3,1)$*-HPMD of type* (5^6) *exists.*

Proof: Let $X = Z_{25} \cup \{\infty_1 \cdots, \infty_5\}$. Let $X_i = \{i, i+5, i+10, i+15, i+20\}$ for $0 \leq i \leq 4$ and $X_5 = \{\infty_1, \ldots, \infty_5\}$. We can construct a (30,3,1)-HPMD of type (5^6) on X having holes $X_0, \ldots, X_5$ and blocks as follows:

$$\begin{array}{lll} (0,3,7) & (\infty_1,0,1) & \\ (0,6,8) & (\infty_2,0,8) & \\ (0,11,9) & (\infty_3,0,9) & \\ (0,12,1) & (\infty_4,0,19) & \\ (0,22,18) & (\infty_5,0,13) & \text{developed mod } 25. \end{array}$$

Lemma 3.9. *A* $(42,3,1)$*-HPMD of type* (7^6) *exists.*

Proof: Let $X = Z_{35} \cup \{\infty_1, \ldots, \infty_7\}$. Let $X_i = \{i, i+5, i+10, i+15, i+20, i+25, i+30\}$ for $0 \leq i \leq 4$ and $X_5 = \{\infty_1, \ldots, \infty_7\}$. We can construct on X a (42,3,1)-HPMD of type (7^6) having holes $X_0, \ldots, X_5$ and blocks as follows:

$$\begin{array}{lll} (0,3,7) & (\infty_1,0,8) & \\ (0,6,8) & (\infty_2,0,9) & \\ (0,11,9) & (\infty_3,0,23) & \\ (0,12,1) & (\infty_4,0,13) & \\ (0,32,28) & (\infty_5,0,14) & \\ (0,16,17) & (\infty_6,0,17) & \\ (0,19,13) & (\infty_7,0,21) & \text{developed mod } 35. \end{array}$$

Proposition 3.10. *For* $m \not\equiv 0 \pmod 3$, *there is a* $(6m,3,\lambda)$*-HPMD of type* (m^6) *except for a nonexisting* $(6,3,1)$*-PMD.*

Proof: When m is even, the conclusion follows from Lemma 3.4. When m is odd and $m \neq 1, 5, 7$, the conclusion follows from Corollary 3.6 and Lemma 3.7. By taking repeated blocks, Lemma 3.8 and Lemma 3.9 will take care of the cases $m = 5$ and 7. When $m = 1$, the conclusion follows from Lemma 2.4.

Combining Propositions 3.1, 3.3 and 3.10, we obtain the main result of this paper.

Theorem 3.11. *An* $(mh,3,\lambda)$*-HPMD of type* (m^h) *exists if and only if*

$$\lambda h(h-1)m^2 \equiv 0 \pmod 3$$

except for a nonexisting $(6,3,1)$*-PMD.*

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