# On Maximal Clique Irreducible Graphs

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**Abstract.** A graph $G$ is said to be maximal clique irreducible if each maximal clique in $G$ contains an edge which is not contained in any other maximal clique of $G$. In 1981, Opsut and Roberts proved that any interval graph is maximal clique irreducible. In this paper we generalize their result and consider the question of characterizing maximal clique irreducible graphs.

## 1. Introduction.

In this paper graphs have no self-adjacent vertices and no multiple edges. Nonempty complete subgraphs of a graph are called *cliques*. *Maximal* cliques of a graph $G$ are those which are contained in no other clique of $G$. We write $m(G)$ to denote the number of maximal cliques of $G$.

A *clique covering* of $G$ is a family $C$ of cliques of $G$ with the property that every edge of $G$ lies in some member of $C$. If a clique covering $C$ has cardinality $|C|$ and $|C'| \geq |C|$ for all clique coverings $C'$ of $G$, then $C$ is called a *minimal* clique covering and $|C|$ is defined to be the clique covering number of $G$, denoted by $cc(G)$.

A clique partition $C$ of $G$ is a clique covering in which every edge belongs to precisely one member of $C$, and a maximal clique partition is a clique partition in which every clique is maximal in $G$. As with clique coverings, a clique partition or maximal clique partition of minimal size is called *minimal*. The size of a minimal maximal clique partition in $G$ is called the maximal clique partition number of $G$, denoted by $mcp(G)$. For a discussion of maximal clique partitions, see [7].

Not every graph has a maximal clique partition, so $mcp(G)$ may not be defined. But, if it is, then clearly

$$cc(G) \leq mcp(G) \leq m(G),$$

with equality when, for example, $G$ is triangle-free. A graph $G$ will be called (maximal clique) irreducible if $cc(G) = m(G)$, otherwise $G$ is said to be (maximal clique) reducible. It is easy to see that a graph $G$ is irreducible if and only if each maximal clique in $G$ contains an edge which is not contained in any other maximal clique of $G$. An irreducible graph $G$ is said to be strict if $mcp(G)$ is defined. We then have the following:

**Proposition 1.1.** *The following statements are equivalent for a given graph $G$:*

(1) *i) $G$ is strictly irreducible;*
(2) *ii) $mcp(G) = m(G)$;*
(3) *iii) $G$ contains no induced subgraph isomorphic to $H_1$ in Figure 1;*
(4) *iv) each edge in $G$ is contained in a unique maximal clique of $G$;*
(5) *v) $mcp(G)$ is defined, and the set of maximal cliques of $G$ is the only minimal clique covering of $G$.*

Proof: iv) $\rightarrow$ v) follows from [6, Remark 2.2]. The proof of the other parts in the order i) $\rightarrow \ldots \rightarrow$ v) $\rightarrow$ i) is easy.

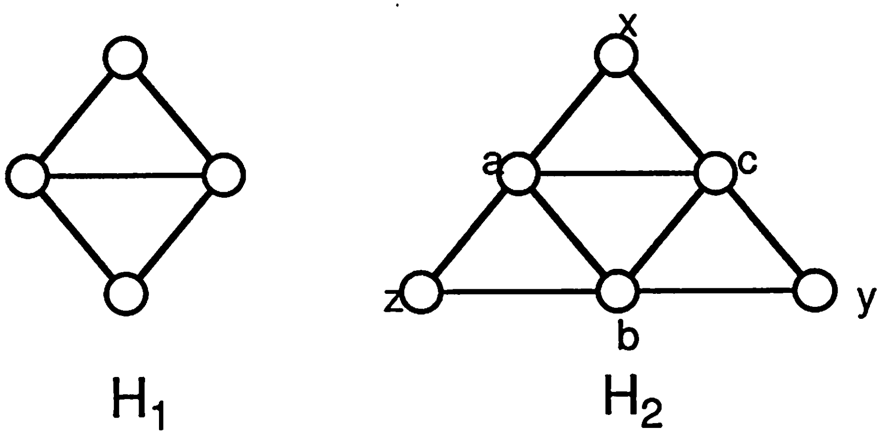


Figure 1

## 2. Irreducible Graphs.

A subgraph of a graph $G$ is called *ocular* if it is isomorphic to the graph $H_2$ shown in Figure 1, where $ay$, $bx$ and $cz$ are not edges of $G$ (but $xy$, $yz$ or $zx$ may or may not be edges of $G$).

We use $M$, possibly with a subscript, to denote a maximal clique and $V(M)$ to denote the set of vertices in $M$. Similarly $C$ (possibly with a subscript) is a clique and $V(C)$ its vertex-set. Clearly a clique is uniquely determined by its set of vertices. We will use the clique $C$ itself to denote its set of vertices provided no confusion can occur. The union of two graphs is defined as usual. We first have the following:

**Theorem 2.1.** *If $G$ is reducible, then $G$ contains an ocular subgraph.*

Proof: Since $cc(G) < m(G)$, we can find a minimum number of maximal cliques in $G$ the union of which contains another maximal clique of $G$, say

$$M \subseteq M_1 \cup M_2 \cup \ldots \cup M_t.$$

Because of the minimality of $t$ we have $M \cap M_1 \not\subseteq M_2 \cup \ldots \cup M_t$. Hence we can find an edge, say $ab$, in $M \cap M_1$ but not in $M_j$ for any $j \geq 2$. We now choose a vertex $c \in V(M)/V(M_1)$ and let $ac \in M_2$, $bc \in M_3$. Clearly $b \notin V(M_2)$ and $a \notin V(M_3)$ (otherwise $M_2$ or $M_3$ would contain the edge $ab$). So $M_1$, $M_2$ and $M_3$ are all different and we can choose a vertex $x \in V(M_2)$ so that $x \not\sim b$ (otherwise $M_2 \cup \{b\}$ would be a clique). Similarly we may select $y \in V(M_3)$ and $z \in V(M_1)$ such that $y \not\sim a$ and $z \not\sim c$. This shows that $G$ contains an ocular subgraph.

We shall use the following result of Fulkerson and Gross [2] :

**Lemma 2.2.** *A graph $G$ is an interval graph if and only if the maximal cliques of $G$ can be ordered as $M_1, M_2, \ldots, M_m$ so that for any vertex $x$ of $G$, if $i < j < k$ and $x \in M_i \cap M_k$, then $x \in M_j$.*

Suppose $G$ contains an ocular subgraph as shown in Figure 1, then by applying Lemma 2.2 and considering the three maximal cliques $M_x$, $M_y$, and $M_z$ containing the triangles $xac$, $ybc$, and $zab$ respectively, we immediately obtain the following:

**Corollary 2.3 (Opsut and Roberts [5]).** *Any interval graph is irreducible.*

Corollary 2.3 implies that Proposition 1.1 is true for an interval graph, which was proved by Ma and Wallis [3] using a different technique.

Given any collection $K$ of cliques in graph $G$, we shall say $K$ has the *Helly property* (as in [1]) if whenever $L_1, L_2, \ldots, L_p$ are in $K$ and $L_i \cap L_j \neq \emptyset$ for all $i, j$ then the total intersection is nonempty, i.e.,

$$\bigcap_{i=1}^{p} L_i \neq \emptyset.$$

We say $K$ has the *strong Helly property* if whenever $L_1, L_2, \ldots, L_p$ are in $K$, then

$$\left|\bigcap_{i=1}^{p} L_i\right| = \min\{|L_i \cap L_j| \,|\, 1 \leq i \neq j \leq p\}.$$

Clearly, $K$ has the strong Helly property if and only if for any three distinct cliques $L_1$, $L_2$ and $L_3$ in $K$,

$$|L_1 \cap L_2 \cap L_3| = \min\{|L_1 \cap L_2|, |L_1 \cap L_3|, |L_2 \cap L_3|\}.$$

This implies that a graph $G$ contains an ocular subgraph if and only if the set of maximal cliques of $G$ does not satisfy the strong Helly property. Hence Theorem 2.1 immediately implies the following:

**Theorem 2.4.** *If the set of maximal cliques in a graph $G$ satisfies the strong Helly property, then $G$ is irreducible.*

It would be interesting to give a characterization of irreducible graphs in terms of the intersection properties of the set of maximal cliques. The following is a characterization of a different kind:

**Theorem 2.5.** *A graph $G$ is reducible if and only if there exists a set of maximal cliques*

$$\mathcal{F} = \{M_1, M_2, \ldots, M_t\}$$

*such that the set of vertices contained in at least two maximal cliques in $\mathcal{F}$ forms a maximal clique different from those in $\mathcal{F}$.*

Proof: The sufficiency is obvious. We now assume that $cc(G) < m(G)$. Let t be the minimum number of maximal cliques in $G$ the union of which contains another maximal clique, say

$$M \subseteq M_1 \cup M_2 \cup \cdots \cup M_t.$$

Define $K$ to be the subgraph of $G$ induced by the set of vertices contained in at least two members of $\mathcal{F} = \{M_1, M_2, \ldots, M_t\}$. Then it suffices to show that $V(M) = V(K)$.

First assume $x \in V(M) \cap V(M_i)$ for some $i$. Then we can select $y \in V(M)/V(M_i)$. Now let $M_j$ be the clique in $\mathcal{F}$ containing the edge $xy$. Then $j \neq i$. Hence $x \in V(M_i) \cap V(M_j) \subseteq V(K)$.

On the other hand, say $x \in V(M_i) \cap V(M_j)$ for some $i$ and $j$ satisfying $1 \leq i \neq j \leq t$ and $x \notin V(M)$. We let $C_0$ be the clique in $G$ such that $V(C_0) = \{x\} \cup (V(M) \cap V(M_i)) \cup (V(M) \cap V(M_j))$, and $M_0$ be a maximal clique containing $C_0$. Then $M \subseteq M_0 \cup \left( \bigcup_{\substack{k \neq i \\ k \neq j}} M_k \right)$, contradicting the minimality of $t$. This completes the proof.

Theorem 2.5 immediately implies the following:

**Corollary 2.6.** *$G$ is irreducible if the following two conditions are satisfied:*

(1) *i) The union of any three maximal cliques in $G$ does not contain another maximal clique of $G$;*

(2) *ii) Given any four maximal cliques in $G$ with an appropriate ordering, say $M_1$, $M_2$, $M_3$ and $M_4$, there exist $x \in V(M_1) \cap V(M_2)$ and $y \in V(M_3) \cap V(M_4)$ such that $x \not\sim y$ in $G$.*

It would be interesting to find irreducible graphs using Corollary 2.6.

## 3. Strictly Irreducible Graphs and Clique Graphs.

Given a graph $G$, let $M_1, \ldots, M_m$ be its maximal cliques. Define a graph $H$ by $V(H) = \{M_i | 1 \leq i \leq m\}$ and $M_i \sim M_j$ in $H$ if and only if $i \neq j$ and $M_i \cap M_j \neq \emptyset$. Then we call $H$ the *clique graph* of $G$, and write $H = G^*$.

We should point out that the definition of a clique graph given above is slightly different from that in [8] in the sense that an isolated vertex is also considered as a maximal clique in [8]. But it is clear that these two definitions are equivalent. The following was obtained by Roberts and Spencer [8]:

**Theorem 3.1.** *A graph $G$ is a clique graph if and only if $G$ contains a clique covering $K$ satisfying the Helly property. Moreover, if $\omega(G) \leq 3$, where $\omega(G) = \max\{|M| \mid M$ is a maximal clique of $G\}$, then $G$ is a clique graph if and only if $G$ has no subgraph isomorphic to the graph of Figure 2.*

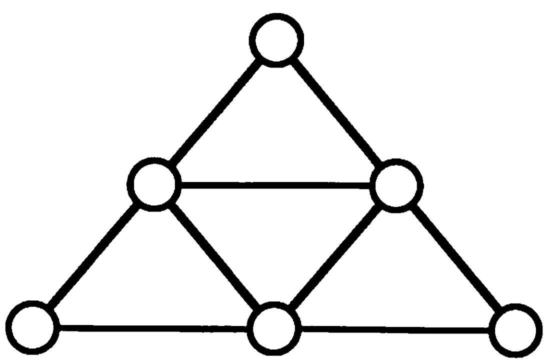

Figure 2

Combining Theorems 2.1 and 3.1 we can easily see the following:

**Corollary 3.2.** *If $\omega(G) \leq 3$, then $G$ is irreducible if and only if $G$ is a clique graph.*

The proof of Theorem 3.1 given in [8] also implies the following:

**Theorem 3.3.** *Every clique covering $\mathcal{F}$ of a graph $H$ satisfying the Helly property induces an irreducible graph $G$ such that $H = G^*$. Moreover $G$ is strict if and only if $\mathcal{F}$ is a clique partition of $H$.*

Remark: Theorem 3.3 implies that a graph $G$ is a clique graph if and only if it is the clique graph of some irreducible graph.

**Theorem 3.4.** *If a graph $G$ is strictly irreducible, then the set of maximal cliques of $G$ satisfies the Helly property.*

Proof: Since $G$ is strictly irreducible, the intersection of any two maximal cliques in $G$ contains at most one element by Proposition 1.1. Let $M_1$ and $M_2$ be two maximal cliques in $G$ such that $M_1 \cap M_2 = \{x\}$. Select any other maximal clique $M$ in $G$ having a nonempty intersection with both $M_1$ and $M_2$. If $x \notin M$, then we may assume $M \cap M_1 = \{y\}$ and $M \cap M_2 = \{z\}$, where $y \notin M_2$ and $z \notin M_1$. Now the maximal clique containing $\{x, y, z\}$ will be different from $M_1, M_2$ and $M$, contradicting the assumption that $G$ is strict. Hence $M \cap M_1 \cap M_2 = \{x\}$, which implies that the set of maximal cliques in $G$ satisfies the Helly property.

**Corollary 3.5.** *A graph $G$ is strictly irreducible if and only if $G$ is the clique graph of some strictly irreducible graph.*

Proof: The necessity follows from Theorem 3.3 and Theorem 3.4. We now assume that $H$ is a strictly irreducible graph, and $G = H^*$. It then suffices to show that $G$ is strictly irreducible.

Select any four maximal cliques $M_1, M_2, M_3$ and $M_4$ in $H$ such that $M_1 \cap M_2 = \{x\}$, $M_3 \cap M_1 \neq \emptyset$, $M_3 \cap M_2 \neq \emptyset$, $M_4 \cap M_1 \neq \emptyset$, and $M_4 \cap M_2 \neq \emptyset$. Then from the proof of Theorem 3.4 we see that $M_3 \cap M_4 = \{x\}$. This implies that $G$ is strictly irreducible by Proposition 1.1.

**Corollary 3.6.** *A graph $G$ is strictly irreducible if and only if $G$ contains a clique partition satisfying the Helly property.*

Notice that the graph $G_1$ in Figure 3 is not a clique graph but is irreducible, while the graph $G_2$ in Figure 3 is a reducible clique graph, and hence $G_2$ is the clique graph of some irreducible graph by the Remark following Theorem 3.3. Therefore, Corollary 3.5 is not true if we delete the word "strict".

Given any graph $G$, let $V_2(G)$ denote the set of vertices of $G$ contained in at least two maximal cliques in $G$. We then have the following interesting result:

**Theorem 3.7.** *Let $G$ be a strictly irreducible graph, and $H$ be the subgraph of $G$ induced by $V_2(G)$. Then $H$ is isomorphic to $G^{**}$, where $G^{**} = (G^*)^*$.*

Proof: For any vertex $x$ in $H$, let $M = \{M_1, \ldots, M_r\}$ be the set of maximal cliques in $G$ containing $x$ where $r \geq 2$. Then from the proof of Theorem 3.4 we can see that $M$ forms a maximal clique of $G^*$, and hence $M$ corresponds to a vertex of $G^{**}$, say $x'$. We then can easily check that the mapping $f: x \rightarrow x'$ from $V(H)$ to $V(G^{**})$ forms an isomorphism between two graphs $H$ and $G^{**}$.

Note added in proof. After this paper was submitted, Prof. McKee pointed out to us that Corollary 3.5 was also obtained in [4].

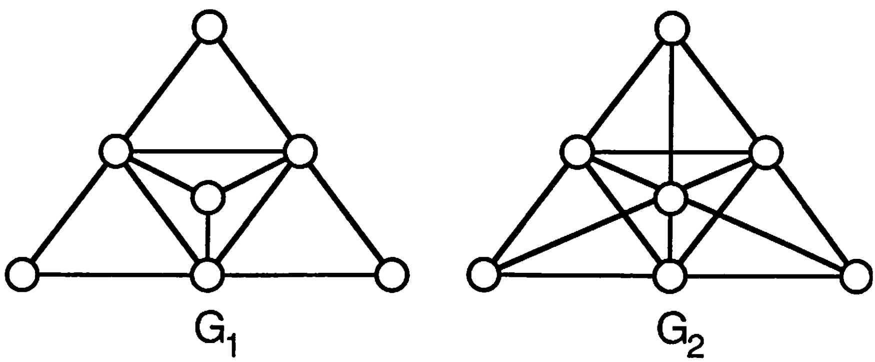


Figure 3

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