A New Approach to Distance Stable Graphs

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Abstract. Let k and ℓ be nonnegative integers not both zero and $D \subseteq N - \{1\}$. A (connected) graph G is defined to be (k, ℓ, D)-stable if for every pair u, v of vertices of G with $d_G(u, v) \in D$ and every set S consisting of at most k vertices of $V(G) - \{u, v\}$ and at most ℓ edges of $E(G)$, the distance between u and v in $G - S$ equals $d_G(u, v)$. For a positive integer m let $N_{\geq m} = \{x \in N \mid x \geq m\}$. It is shown that a graph is $(k, \ell, \{m\})$-stable if and only if it is $(k, \ell, N_{\geq m})$-stable. Further, it is established that for every positive integer x, a graph is $(k + x, \ell, \{2\})$-stable if and only if it is $(k, \ell + x, \{2\})$-stable. A generalization of $(k, \ell, \{m\})$-stable graphs is considered. For a planar $(k, 0, \{m\})$-stable graph, $m \geq 3$, a sharp bound for k in terms of m is derived.

1. Characterizations of distance stable graphs.

(Graph Theory terminology not presented here appears in [3].) Ali, Boals and Sherwani [1] defined a connected graph G to be *vertex (edge) distance stable* if the distance between nonadjacent vertices is unchanged after the deletion of a vertex (edge) of G. They showed that a graph is vertex distance stable if and only if it is edge distance stable. Thus, for convenience, we refer to vertex or edge distance stable graphs simply as distance stable graphs. Further, distance stable graphs are characterized as those which satisfy for every pair u, v of nonadjacent vertices, $N(u) \cap N(v) = \emptyset$ or $|N(u) \cap N(v)| \geq 2$. This characterization thus suggests an efficient way of determining whether a graph is distance stable. Moreover, this characterization of distance stable graphs has the following alternative statement. A graph G is distance stable if and only if every pair of vertices at distance 2 apart is connected by at least two internally disjoint paths, or equivalently, at least two edge disjoint paths of length 2. Intuitively this result implies that if "short" distances between pairs of vertices (namely those at distance 2) are preserved after the deletion of a vertex or edge, then so are "longer" distances between pairs of vertices.

Observe that distance stable graphs are certainly 2-connected, since the deletion of a vertex in such a graph does not destroy the property of connectedness between pairs of nonadjacent vertices. However, the property of being distance stable is much stronger than the property of being 2-connected. For example, every cycle of length at least 5 is 2-connected but not distance stable. This observation suggests a generalization of distance stable graphs. A connected graph G is *k-vertex (edge) distance stable* if the distance between (nonadjacent) vertices is unchanged after the deletion of any set at most k vertices (edges). The following result provides a simple characterization of these graphs. We use $d_G(u,v)$ to denote the distance between vertices u and v in a graph G.

Proposition 1. *For a graph G and integer $k \geq 1$ the following are equivalent:*

(i) *G is k-vertex distance stable.*
(ii) *G is k-edge distance stable.*
(iii) *For every pair u, v of nonadjacent vertices of G either $N(u) \cap N(v) = \emptyset$ or $|N(u) \cap N(v)| \geq k+1$.*
(iv) *For every pair u, v of nonadjacent vertices of G there exist $k+1$ internally disjoint $u-v$ paths of length $d_G(u,v)$.*

Proof: (i) $\Rightarrow$ (ii). Suppose G is k-vertex distance stable. Let $E = \{e_1, e_2, \ldots, e_k\}$ be any set of k edges of G. Let u and v be any two nonadjacent vertices of G. For each i $(1 \leq i \leq k)$ let v_i be a vertex different from u and v that is incident with e_i, and let $V = \{v_i \mid 1 \leq i \leq k\}$. Since G is k-vertex distance stable, $d_{G-V}(u,v) = d_G(u,v)$. Also since $G - V \subset G - E$, $d_{G-V}(u,v) \geq d_{G-E}(u,v) \geq d_G(u,v)$. So $d_{G-E}(u,v) = d_G(u,v)$.

(ii) $\Rightarrow$ (iii). Suppose G is k-edge distance stable. Let u and v be a pair of nonadjacent vertices with $N(u) \cap N(v) = \emptyset$. Suppose $|N(u) \cap N(v)| \leq k$. Let $\{v_1, v_2, \ldots, v_n\} = N(u) \cap N(v)$ and $E = \{uv_i \mid 1 \leq i \leq n\}$. Then $|E| \leq k$ and $d_{G-E}(u,v) > d_G(u,v) = 2$, contrary to assumption. So if G is k-edge distance stable, then $|N(u) \cap N(v)| \geq k+1$ for every pair u, v of vertices at distance 2 apart in G.

(iii) $\Rightarrow$ (i). Let u, v be any pair of nonadjacent vertices of G and V a subset of at most k vertices in $V(G) - \{u, v\}$. Let $d = d_G(u,v)$. Assuming (iii) we prove by induction on d that $d_{G-V}(u,v) = d_G(u,v)$. If $d = 2$, then the result follows since $|N(u) \cap N(v)| \leq k+1$. Suppose now that $d \geq 3$ and that for every subset W of at most k vertices and for every pair x, y of nonadjacent vertices of $G - W$ with $d_G(x,y) = d-1$, we have $d_{G-W}(x,y) = d_G(x,y) = d-1$. Let u, v be a pair of nonadjacent vertices in $G - V$ with $d_G(u,v) = d$. Let $P\colon u = u_1, u_2, \ldots, u_{d+1} = v$ be a shortest $u-v$ path in G.

Let $V' = V - \{u_2\}$. Then V' contains at most k vertices. Since $d_G(u_2, u_{d+1}) = d-1$, our inductive hypothesis implies that $G - V'$ contains a $u_2 - u_{d+1}$ path P' of length $d-1$. Observe that every vertex of P' different from u_2 also belongs to $G - V$. Let u' be the vertex that precedes u_{d+1} on P'. Then $d_G(u, v') = d-1$. So by our inductive hypothesis $G - V$ contains a $u - u'$ path Q' of length $d-1$. Thus Q' followed by the path u', u_{d+1} is a $u-v$ path of length d in $G - V$. Thus $d_{G-V}(u,v) = d$.

Clearly (iv) $\Rightarrow$ (i). We now show that (i) $\Rightarrow$ (iv). Suppose there exists a pair u, $v \in V(G)$ with $d_G(u,v) = d \geq 2$ such that G has at most k internally disjoint $u - v$ paths of length d. Let G' be the directed graph obtained from all $u - v$ paths of length d in G by assigning an edge xy the direction (x, y) if x precedes y on a shortest $u - v$ path of G. Then the maximum number of internally disjoint directed $u - v$ paths in G' is $t \leq k$, since every directed $u - v$ path in G' must have length d. Thus by Menger's theorem there exists a set S of t vertices whose removal separates u and v in G'. Thus $G - S$ contains no $u - v$ path of length d. This implies that G is not k-vertex distance stable. ∎

As remarked earlier, every vertex distance stable graph is 2-connected. Similarly every k-vertex distance stable graph is $(k+1)$-connected; but the converse does not hold. For example, let G be the graph obtained from $p \geq 5$ copies of K_k, denoted by $H_1, H_2, \ldots, H_p$, by joining every vertex of H_i to every vertex of H_{i+1} ($1 \leq i \leq p$ and subscripts expressed modulo p). Then G is $(k+1)$-connected, but G is not k-vertex distance stable. Note that all complete graphs are k-vertex distance stable for all $k \in \boldsymbol{N}$. Therefore we now consider graphs that are not complete.

The connectivity (edge-connectivity) of a noncomplete connected graph G is a measure that determines the maximum integer $\kappa(\lambda)$ such that after the removal of any set of fewer than κ vertices (λ edges) every remaining pair of vertices is still connected by a path. Since $(k+1)$-connected graphs need not be k-vertex distance stable, this observation suggests the study of a new parameter. The *vertex-deletion (edge-deletion) distance stability* $\kappa_s(G)(\lambda_s(G))$ of a graph G is the maximum integer κ_s (λ_s) such that after the removal of any set with fewer than κ_s vertices (λ_s edges) distances between nonadjacent vertices are still preserved. By Proposition 1, we need only determine $|N(u) \cap N(v)|$ for all pairs of vertices at distance 2 apart. Then both $\kappa_s(G)$ and $\lambda_s(G)$ equal the minimum overall these quantities, that is,

$$\kappa_s(G) = \lambda_s(G) = \min\{|N(u) \cap N(v)| \mid u, v \in V(G) \text{ and } d_G(u,v) = 2\}.$$

Thus both $\kappa_s(G)$ and $\lambda_s(G)$ can be computed efficiently.

In 1967 Beineke and Harary [2] pointed out that up to that time the problem of disconnecting a pair of vertices and thus a graph by deleting a combination of vertices and edges had been overlooked. So they defined for a given G, with connectivity κ, the connectivity function $f: \{0, 1, \ldots, k\} \rightarrow \boldsymbol{N} \cup \{0\}$ as follows:

$$f(k) = \min\{\lambda(G - S) \mid S \subseteq V(G) \text{ and } |S| = k\}$$

and provided a characterization of these functions.

These concepts of the connectivity function of a graph and vertex distance and edge distance stability of a graph suggest another measure of "distance stability" in a graph. We say that a graph is k-vertex ℓ-edge distance stable if for every set S with at most k vertices and at most ℓ edges and every pair u, v of nonadjacent vertices of $G - S$, we have $d_{G-S}(u,v) = d_G(u,v)$. The next result gives a characterization of k-vertex ℓ-edge distance stable graphs.

Proposition 2. *Let G be a graph and k and ℓ nonnegative integers with $k+\ell \geq 1$. Then the following are equivalent:*

(i) *G is $(k+\ell)$-vertex distance stable.*
(ii) *G is $(k+\ell)$-edge distance stable.*
(iii) *G is k-vertex ℓ-edge distance stable.*

Proof: We have already shown that (i) $\Leftrightarrow$ (ii). To complete the proof we need only observe as in the proof of Proposition 1 that (i) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (ii). ∎

The results of Propositions 1 and 2 depend on the fact that not all shortest paths between pairs of vertices at distance 2 apart are destroyed by the deletion of a specified number of vertices and/or edges. Since edge-disjoint paths of length at most 2 between a pair of vertices are also internally disjoint and vice versa, the equivalence of k-vertex distance stable graphs and k-edge distance stable graphs is less surprising. With these observations in mind we consider yet another interpretation and extension of distance stable graphs. Let k and ℓ be nonnegative integers not both 0 and suppose $D \subseteq \mathbf{N} - \{1\}$. We say that a graph is k-vertex ℓ-edge D-distance stable, denoted by (k,ℓ,D)-stable if for every set S of at most k vertices and at most ℓ edges and every pair u, v of vertices in $G - S$ with $d_G(u,v) \in D$ we have $d_{G-S}(u,v) = d_G(u,v)$. For $m \geq 2$ an integer let $N_{\geq m} = \{m, m+1, m+2, \ldots,\}$. Then the following result is an immediate consequence of Propositions 1 and 2.

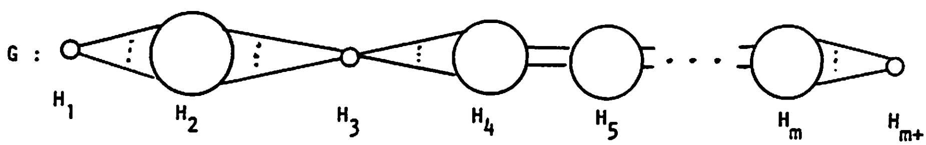
Corollary. *For a graph G and nonnegative integers k and ℓ with $k+\ell \geq 1$, the following are equivalent:*

(i) *G is $(k+\ell, 0, \{2\})$-stable.*
(ii) *G is $(0, k+\ell, \{2\})$-stable.*
(iii) *G is $(k+\ell, 0, N_{\geq 2})$-stable.*
(iv) *G is $(0, k+\ell, N_{\geq 2})$-stable.*
(v) *G is $(k, \ell, N_{\geq 2})$-stable.*

If $2 \in D$, then it can be shown as in the proof of Proposition 1 that a graph G is $(k+x, \ell, D)$-stable if and only if G is $(k, \ell+x, D)$-stable. Observe that if a graph is $(\ell, 0, \{3\})$-stable, then it is also $(0, \ell, \{3\})$-stable. Suppose now that G is $(0, \ell, \{3\})$-stable. Assume G is not $(\ell, 0, \{3\})$-stable. Then there exists a pair u, v of vertices at distance 3 in G and a set of at most ℓ vertices whose removal increases the distance between u and v. Let S be such a set with the smallest number of vertices. Then $S \subseteq N(u) \cup N(v)$. However, then the removal of $\{ux \mid x \in N(u) \cap S\} \cup \{yv \mid y \in N(v) \cap S\}$ also increases the distance from

u to v. Hence G is not $(0,\ell,\{3\})$-stable. Thus if a graph is $(0,\ell,\{3\})$-stable, then it is $(\ell,0,\{3\})$-stable.

However, if $m \geq 4$ and a graph is $(0,\ell,\{m\})$-stable, then it need not be $(\ell,0,\{m\})$-stable, in fact not even $(1,0,\{m\})$-stable. Let $H_1 \cong H_3 \cong H_{m+1} \cong K_1$ and for $2 \leq i \leq m$ and $i \neq 3$ let $H_i \cong K_{\ell+1}$. Now define G to be the graph obtained from $\bigcup_{j=1}^{m+1} H_j$ by joining each vertex of H_i to every vertex of H_{i+1} for $1 \leq i \leq m$, (see Figure 1). Then G is $(0,\ell,\{m\})$-stable but not $(1,0,\{m\})$-stable.



However, we have the following:

Proposition 3. *For integers $m \geq 2$, $k \geq 0$, $\ell \geq 0$ where $k+\ell \geq 1$ and a graph G the following are equivalent:*

(i) *G is $(k,\ell,\{m\})$-stable.*
(ii) *G is $(k,\ell,N_{\geq m})$-stable.*

Proof: Clearly (ii) $\Rightarrow$ (i). Suppose now that G is $(k,\ell,\{m\})$-stable. Let S be any set of k vertices and ℓ edges and let u, v be vertices of $G-S$ such that $d_G(u,v) = d \geq m$. We prove by induction on d that $d_{G-S}(u,v) = d$. If $d = m$, then the result follows from the hypothesis. Suppose thus that $d > m$ and that whenever T is a set of at most k vertices and at most ℓ edges and $x, y \in V(G)-T$ such that $d_G(x,y) = d-1$, then $d_{G-T}(x,y) = d_G(x,y)$. Let $P\colon u = u_1, u_2, \ldots, u_d, u_{d+1} = v$ be a (shortest) $u-v$ path in G. Then $d_G(u_2,v) = d-1$. So if $S' = S - \{u_2\}$, then by the inductive hypothesis $d_{G-S'}(u_2,v) = d_G(u_2,v) = d-1$. Let Q be a $u_2 - v$ path in $G-S'$ of length $d-1$. Let u' be the vertex that precedes v on this path. Then $d_G(u_1,u') = d-1$ and $u' \notin S$. So by the inductive hypothesis there is a $u_1 - u'$ path Q' of length $d-1$ in $G-S$. Then Q' followed by the path u', v is a $u-v$ path of length d in $G-S$. ∎

If $\emptyset \neq D \subseteq N - \{1\}$ and m is the smallest element of D, then Proposition 3 implies that a graph is $(k,\ell,\{m\})$-stable if and only if it is (k,ℓ,D)-stable. We now focus our attention on $(k,\ell,\{m\})$-stable graphs where $m \geq 2$ is an integer. To determine the maximum k for which a graph G is $(k,0,\{m\})$-stable, first determine all pairs of vertices at distance m apart. Suppose u, v is a pair of vertices such that $d_G(u,v) = m$. Then a breadth-first search can be used to determine the subgraph induced by all those vertices that lie on some shortest $u-v$ path. Denote this subgraph by $G(u,v)$. Let $\kappa(u,v)$ be the smallest number of vertices whose removal disconnects u and v in $G(u,v)$. Then $k = \min\{\kappa(u,v) - 1 \mid u, v \in V(G)$ and $d_G(u,v) = m\}$. Thus the maximum k for which G is $(k, 0, \{m\})$-stable can be computed efficiently. Similarly the maximum ℓ for which G is $(0, \ell, \{m\})$-stable can be found efficiently.

Thus far we have been concerned with graphs that preserve distances between pairs of vertices, a given distance apart, after the deletion of any set that contains at most a given number of vertices and/or edges. In particular, if G is (k, ℓ, D)-stable $u, v \in V(G)$ with $d_G(u,v) \in D$ and S is a set of at most k vertices of $V(G) - \{u,v\}$ and at most ℓ edges of G, then $G - S$ contain at least one $u - v$ path of length $d_G(u,v)$. We may well wish to require that $G - S$ contains r internally disjoint $u - v$ paths of length $d_G(u,v)$ for some r at least 2. With this in mind suppose k, ℓ, and r are integers with $k + \ell \geq 1$ and $r \geq 1$ and that $D \geq N - \{1\}$. Then we say that a graph G is (k, ℓ, D, r)-stable if for any pair $u, v \in V(G)$ such that $d_G(u,v) \in D$ and any set S of at most k vertices of $G - \{u,v\}$ and at most ℓ edges of G, the graph $G - S$ contains at least r internally disjoint $u - v$ paths of length $d_G(u,v)$. Thus a graph is (k, ℓ, D)-stable if and only if it is $(k, \ell, D, 1)$-stable. The next result shows however that (k, ℓ, D, r)-stable graphs have already been studied.

Proposition 4. *Let k, ℓ and r be integers with $k + \ell \geq 1$ and $r \geq 1$ and let $D \subseteq N - \{1\}$. Then G is (k, ℓ, D, r)-stable if and only if it is $(k + r - 1, \ell, D)$-stable.*

The proof is omitted since it is a simple generalization of the last part of the proof in Proposition 1.

Remark: By applying techniques similar to those employed in the proofs of Propositions 1, 2, 3 and 4, analogues of these results can be obtained for directed graphs.

2. Planar distance stable graphs.

In this section we study planar (k, ℓ, D)-stable graphs. If $\phi \neq D \subseteq N - \{1\}$ and m is the smallest element of D, then it follows from Proposition 3 that a graph G is (k, ℓ, D)-stable if and only if G is $(k, \ell, \{m\})$-stable. Thus when these graphs are studied it suffices to consider pairs of vertices distance m apart. We focus our attention on the diameter of planar $(k, 0, \{m\})$-stable graphs and relationships between k and m for these graphs.

Consider first planar 1-vertex distance stable graphs. Observe that C_4 is a planar 1-vertex distance stable graph with diameter 2. Let $d \geq 3$ and let $H_1, H_2, \ldots,$ H_{d-2} be $d - 2$ copies of K_4 with $V(H_i) = \{u_i, v_i, x_i, w_i\}$. Let G_d be obtained from $H_1, H_2, \ldots, H_{d-2}$ by identifying for $1 \leq i < d - 2$ the vertices x_i and w_i with u_{i+1} and v_{i+1}, respectively, and then joining a new vertex u to u_1 and v_1 and another vertex v to x_{d-2} and w_{d-2}. Figure 2 shows G_5. Note that G_d is a planar 1-vertex distance stable graph having diameter d; so in particular, G_d is $(1, 0, \{m\})$-stable, for $2 \leq m \leq d$.

G_5:

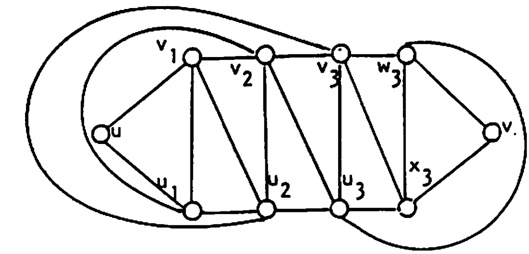


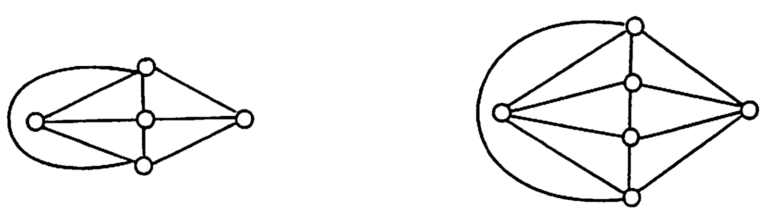
Figure 2

The next result shows that if $k \geq 2$, then the diameter of a k-vertex distance stable graph cannot be too large.

Proposition 5. *If G is a k-vertex distance stable graph where $k \geq 2$, then diam $G \leq 2$.*

Proof: Suppose diam $G \geq 3$. Then G contains a pair of vertices u, w such that $d_G(u, w) = 3$. Let v be a vertex that precedes w on a shortest $u - w$ path. Then $d_G(u, v) = 2$. Since G is k-vertex distance stable, there exist at least $k + 1 \geq 3$ internally disjoint $u - v$ paths of length 2. Let x, y and z be three vertices in $N(u) \cap N(v)$. We may assume that y and w lie on opposite sides of the cycle u, x, v, z, u in a plane embedding of G, otherwise w and x lie on opposite sides of the cycle u, y, v, z, y or w and z lie on opposite sides of the cycle u, x, v, y, u. Since $d_G(y, w) = 2$ there exists at least one $y - w$ path of length 2 different from y, v, w, that contains either x or z. But then x or z is adjacent with w, which is not possible since $d_G(u, w) = 3$. So diam $G \leq 2$. ∎

Figure 3 shows k-vertex distance stable graphs for $k = 2, 3$, respectively.



A 2-vertex distance stable graph

A 3-vertex distance stable graph

Figure 3

Since a planar graph G has $\delta(G) \leq 5$, it follows that if G is a planar k-vertex distance stable graph, then $k + 1 \leq \delta(G) \leq 5$, that is, $k \leq 4$. We show next that there is no planar 4-vertex distance stable graph. If G is a planar 4-vertex distance stable graph, then there exists a pair u, v of nonadjacent vertices such that $|N(u) \cap N(v)| \geq 5$. Let $v_1, v_2, \ldots, v_5$ be vertices in $N(u) \cap N(v)$ and assume that this is the order in which they appear in a plane embedding of G as we proceed

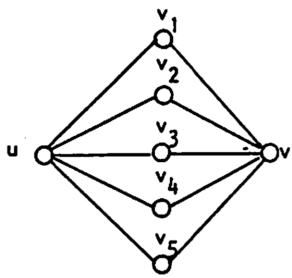


Figure 4

in counter-clockwise order about v. Then the plane embedding of G contains the plane subgraph of Figure 4. Clearly $d_G(v_1, v_4) = 2$, but $|N(v_1) \cap N(v_4)| \leq 3$.

We have thus shown that there exist planar k-vertex distance stable graphs if and only if $k = 1, 2$ or 3. Next we consider $(k, 0, \{m\})$-stable graphs for $m \geq 3$.

Proposition 6. *Let $m \geq 3$ be an integer. Then there exists a planar $(k, 0, \{m\})$-stable graph if $1 \leq k \leq 2m-3$ or if $m = 3$ and $k = 2m-2$.*

Proof: If $m = 3$ and $k = 2m - 2$, then the icosahedron shown in Figure 5 is $(k, 0, \{m\})$-stable.

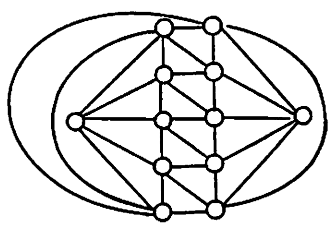


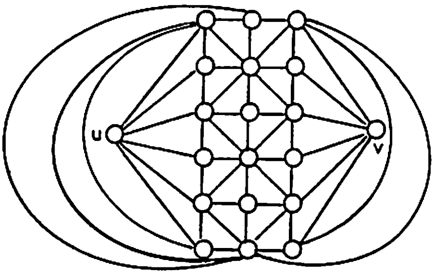
Figure 5

Suppose now that $m \geq 3$ and $1 \leq k \leq 2m-3$. We show that there is a $(k, 0, \{m\})$-stable graph by first constructing a $(2m-3, 0, \{m\})$-stable graph.

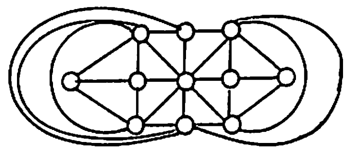
For $j = 0, 1, \ldots, 2m-3$ let H_j be the path $u, v_{j,1}, v_{j,2}, \ldots, v_{j,m-1}, v$. We now construct a graph $G_{2m-3,m}$ from $H_0, H_1, \ldots, H_{2m-3}$ by first identifying the vertex u from each H_j in a vertex labelled u and identifying the vertex v from each H_j in a vertex labelled v. Next add for every i $(1 \leq i \leq m-1)$ the edges of $\{v_{\ell,i}v_{\ell+1,i} \mid 0 \leq \ell \leq 2m-3$ and where $\ell + 1$ is expressed modulo $2m-2\}$. Finally add the edges of $\{v_{j,i}v_{j+1,i+1}, v_{j,i}v_{j-1,i+1} \mid$ where j is even and $0 \leq j < 2m-3$ and the subscripts $j+1$ and $j-1$ are expressed modulo $2m-2$ and i is odd with $1 \leq i \leq m-2\} \cup \{v_{j,i}v_{j+1,i+1}, v_{j,i}v_{j-1,i+1}$ where j is odd and $1 \leq j \leq 2m-3$ and $j+1$ and $j-1$ are expressed modulo $2m-2$ and i is even and $2 \leq i \leq m-2\}$ to produce $G_{2m-3,m}$. It can be shown that

$G_{2m-3,m}$ is a planar $(2m-3, 0, \{m\})$-stable graph. Suppose now that k is a positive integer with $k = (2m-3) - r$ for some positive integer r. Let $G_{k,m}$ be obtained from $G_{2m-3,m}$ by identifying for each i $(1 \le i \le m-1)$ the vertices of the set $\{v_{2m-3,i}, v_{2m-4,i}, \dots, v_{2m-3-r,i}\}$ in a vertex labelled $v_{2m-3-r,i}$. By observing that the eccentricity of every vertex other than u or v is at most $m-1$ we see that $G_{k,m}$ is a planar $(k, 0, \{m\})$-stable graph. ∎

Figure 6 shows $G_{5,4}$ and $G_{2,4}$.



$G_{5,4}$



$G_{2,4}$

Figure 6

The next results shows that if G is a planar $(k, 0, \{m\})$-stable graph with diam $G \ge m$, then k cannot be too large in comparison to m.

Proposition 7. *Let $m \ge 3$ be an integer and G a planar $(k, 0, \{m\})$-stable graph with diam $G \ge m$. Then $1 \le k \le 2m-2$ if $m = 3$ and $1 \le k \le 2m-3$ for $m \ge 4$.*

Proof: Let G be a planar $(k, 0, \{m\})$-stable graph with diam $G \ge m \ge 3$. Let u, v be a pair of vertices with $d_G(u, v) = m$. Then there exist at least $k+1$ internally disjoint $u-v$ paths in G. Let H_i: $u, v_{i,1}, v_{i,2}, \dots, v_{i,m-1} v$, $0 \le i \le k$ be $k+1$ internally disjoint $u-v$ paths in G. Consider a plane embedding of G and suppose that the paths $H_0, H_1, \dots, H_k$ have been labelled in the order in which they emanate from u as we proceed about u in the clockwise direction. Observe that $d_G(v_{i,1}, v_{j,m-1}) \le m$ for $0 \le i, j \le k$. If $d_G(v_{i,1}, v_{j,m-1}) = m$, then G contains at least $k+1$ internally disjoint $v_{i,1} - v_{j,m-1}$ paths.

We show first that $k \le 2m-2$. If $k \le 2m-1$, then there exist at least $2m$ internally disjoint $u-v$ paths. Consider the vertex $v_{1,1}$. If $d_G(v_{1,1}, v_{j,m-1}) = m$, for some $j \ge 3$, then there exist at least $2m$ internally disjoint $v_{1,1} - v_{j,m-1}$ paths in G. Hence each of these paths must contain one and only one of the $2m$ vertices $u, v_{0,1}, v_{0,2}, \dots, v_{0,m-1}, v, v_{2,m-1}, v_{2,m-2}, \dots, v_{2,1}$. Further, one of these paths must contain $v_{2,m-1}$ and another one $v_{0,m-1}$. Since $d_G(v_{1,1}, v_{0,m-1}) \ge m-2$ and $d_G(v_{1,1}, v_{2,m-1}) \ge m-2$, it follows that $j = 3$ or 4 and $j = k$ or $k-1$. Since $m \ge 3$, $k-1 \ge 2m-2 \ge 4$. So $j = 4$ and $m = 3$. Observe that in this case there cannot exist $2m$ internally disjoint $v_{1,m-1} - v_{j,1}$ paths. Thus $d_G(v_{1,m-1}, v_{j,1}) \leq m - 1 = 2$. However, it is not difficult to see that $d_G(v_{1,m-1}, v_{4,1}) > 2$. So this situation cannot occur. Therefore $d_G(v_{1,1}, v_{j,m-1}) \leq m - 1$ for $3 \leq j \leq 2m - 1 = 5$. However, once again it can be shown that $d_G(v_{1,1}, v_{m+1,m-1}) > m - 1$. Hence $k \leq 2m - 2$.

We show next that if $m \geq 4$, then $k < 2m - 2$. Suppose $k = 2m - 2$. Then there exist $2m - 1$ internally disjoint $u - v$ paths in G. Consider the vertex $v_{1,1}$. If $d_G(v_{1,1}, v_{j,m-1}) = m$ for some $j \geq 3$, then there exists a collection C of at least $2m - 1$ internally disjoint $v_{1,1} - v_{j,m-1}$ paths of length m and each of these paths must contain at least one of the $2m$ vertices on the cycle C: $u, v_{0,1}, v_{0,2}, \dots, v_{0,m-1}, v, v_{2,m-1}, v_{2,m-2}, \dots, v_{2,1}$. Further, at most one of these paths contains two vertices of C. Suppose now that C contains two distinct paths R and S such that R contains $v_{2,m-1}$ and S contains $v_{0,m-1}$. If neither the $v_{2,m-1} - v_{j,m-1}$ path of R nor the $v_{0,m-1} - v_{j,m-1}$ path of S contains v, then $3 \leq j \leq 4$ and $2m - 3 \leq j \leq 2m - 2$. However, this is impossible, since $m \geq 4$. Clearly C contains either a $v_{1,1} - v_{j,m-1}$ path that contains $v_{2,m-1}$ and no other vertex of C or a $v_{1,1} - v_{j,m-1}$ path that contains $v_{0,m-1}$ and no other vertex of C. Suppose the former occurs and let P be such a $v_{1,1} - v_{j,m-1}$ path. (The argument for the second case is similar.) Then necessarily another one of the paths of C contains $v_{0,m-2}$ but no other vertex of the cycle C, otherwise, as before, a contradiction is produced. So $j = 3$ or 4 and $j = 2m - 2, 2m - 3$ or $2m - 4$. Since $m \geq 4$, this implies $j = 4$ and $m = 4$ and that $k = 2m - 2 = 6$. So $v_{1,1}v_{0,2}$ is an edge and the $v_{1,1} - v_{4,3}$ path P contains a $v_{1,1} - v_{2,3}$ path P' of length 2, say $v_{1,1}, x, v_{2,3}$. (See Figure 7.) Notice that $v_{1,1}, v_{0,2}, v_{0,3}, v, v_{2,3}$ and P' forms a cycle of length 6 that contains $v_{1,3}$ in its interior region. Since $k = 6$, it now follows that $d_G(v_{1,3}v_{j,1}) < 4$ for $j = 2, 3, 4, 5$ and 6. In particular $d_G(v_{1,3}, v_{4,1}) \leq 3$. It is not difficult to see that $d_G(v_{1,3}, v_{4,1}) \geq 3$. So $d_G(v_{1,3}, v_{4,1}) = 3$. Since $x \neq v_{2,2}$, it follows that $v_{1,3}v_{2,2} \notin E(G)$. Therefore the only path of length 3 from $v_{1,3}$ to $v_{4,1}$ is $v_{1,3}, v_{2,3}, v_{3,2}, v_{4,1}$.

Observe that $u, v_{1,1}, x, v_{2,3}, v_{3,2}, v_{4,1}, u$ is a 6-cycle that contains $v_{3,1}$ in its interior. So $d(v_{3,1}, v_{j,3}) \leq 3$ for $0 \leq j \leq 6$. In particular $d(v_{3,1}, v_{6,3}) \leq 3$. However, $d(v_{3,1}, v_{6,3}) \geq 3$. So $d(v_{3,1}, v_{6,3}) = 3$. Since $v_{3,2}, v_{4,1} \in E(G)$, it follows that the only path of length 3 from $v_{3,1}$ to $v_{6,3}$ is $v_{3,1}, v_{4,1}, v_{5,2}, v_{6,3}$. However, now $v, v_{6,3}, v_{5,2}, v_{4,1}, v_{3,2}v_{2,3}, v$ is a 6-cycle that contains $v_{4,3}$ in its interior. This is impossible, since there are seven internally disjoint $v_{1,1} - v_{4,3}$ paths. This shows that $d_G(v_{1,1}, v_{j,m-1}) \leq m - 1$ for all j $(0 \leq j \leq 2m - 2)$. Similarly $d_G(v_{i,1}, v_{j,m-1}) \leq m - 1$ for all i and j $(0 \leq i, j \leq 2m - 2)$.

Suppose Q is some $v_{r,1} - v_{s,m-1}$-path. If $v_{i,j}, v_{i+1,j+1}$ (or $v_{i,j}, v_{i-1,j-1}$) is a path of Q as we proceed from $v_{r,1}$ to $v_{s,m-1}$, then this path will be called an upward diagonal move to the right (or left) (where $1 \leq j \leq m - 2$ and $0 \leq i \leq 2m - 2$ and $i \pm 1$ is expressed modulo $2m - 1$). If Q contains a path of the type $v_{i,j}, v_{i+1,j}$ (or $v_{i,j}, v_{i-1,j}$), as we proceed from $v_{r,1}$ to $v_{s,m-1}$, then such a path will be called

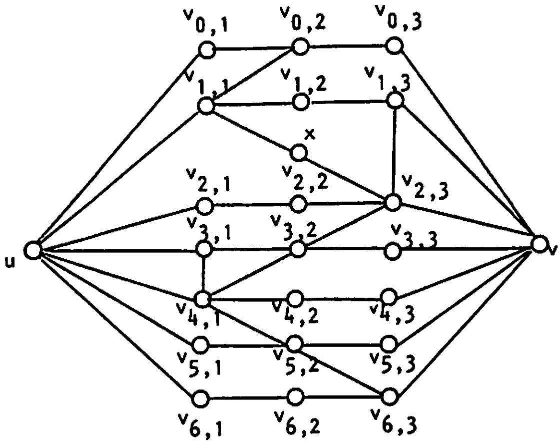


Figure 7

a horizontal move to the right (left).

From the case we are considering it follows that $d_G(v_{i,1}, v_{i\pm(m-1),m-1}) \leq m-1$ for all $0 \leq i \leq 2m-2$ and where $i\pm(m-1)$ is expressed modulo $2m-2$. (We assume henceforth that all first subscripts are expressed modulo $2m-2$.) From the planar embedding of G we are considering it follows that $d_G(v_{i,1} - v_{i\pm(m-1)}) \geq m-1$. So $d_G(v_{i,1}, v_{i\pm(m-1),m-1}) = m-1$. If Q is a $v_{i,1} - v_{i+(m-1),m-1}$ path (or $v_{i,1} - v_{i-(m-1),m-1}$ path) of length $m-1$, then observe that Q contains $m-2$ upward diagonal moves to the right (left) and one horizontal move to the right (left).

We show first that the horizontal move in a $v_{i,1} - v_{i+m-1,m-1}$ path Q, of length $m-1$, is either $v_{i,1}, v_{i+1,1}$ or $v_{i+m-2,m-1}, v_{i+m-1,m-1}$. If this is not the case, then the horizontal move is of the type $v_{j,\ell}, v_{j+1,\ell}$ for some j $(i < j < i+m-2)$. So $v_{i+m-2,m-2}, v_{i+m-1,m-1}$ is an upward diagonal move to the right in Q_1. This forces the horizontal move to the left in a $v_{i+2m-3,1} - v_{i+-m-2,m-1}$ path of length $m-1$ to be $v_{i+m-1,m-1}, v_{i+m-2,m-1}$. Therefore $v_{i+m,m-2}, v_{i+m-1,m-1}$ is an upward diagonal move to the left in every $v_{i+2m-3,1} - v_{i+m-2,m-1}$ path of length $m-1$. If we now consider a $v_{i+1,1} - v_{i+m,m-1}$ path Q_2 of length $m-1$, it can be seen that the horizontal move to the right in Q_2 is $v_{i+m-1,m-1}, v_{i+m,m-1}$. So $v_{i+1,1}, v_{i+2,2}$ is an upward diagonal move to the right in Q_2. This forces the horizontal move to the left in a $v_{i+2,1} - v_{i-m+3,m-1}$ path Q_3 of length $m-1$ to be $v_{i+2,1}, v_{i+1,1}$. So $v_{i+1,1}, v_{i,2}$ is an upward diagonal move to the left in Q_3. This is not possible since $v_{i,1}, v_{i+1,2}$ is an upward diagonal move to the right in Q_3.

Therefore the horizontal move to the right in a $v_{i,1} - v_{i+m-1,1}$ path of length $m-1$ is either $v_{i,1}, v_{i+1,1}$ or $v_{i+m-2,m-1}, v_{i+m-1,m-1}$. Similarly the horizontal move to the left in a $v_{i,1} - v_{i-m+1,m-1}$ path of length $m-1$ in G is either $v_{i,1}, v_{i-1,1}$ or $v_{i-m+2,m-1}, v_{i-m+1,m-1}$. The above argument also implies that for every i $(0 \leq i \leq 2m-2)$ not both $v_{i,1}, v_{i+1,2}$ and $v_{i+1,1}, v_{i+2,2}$ are edges of G. Similarly not both $v_{i,m-2}, v_{i+1,m-1}$ and $v_{i+1,m-2}, v_{i+2,m-1}$ are edges of G.

Consider a $v_{i,1} - v_{i+m-1,m-1}$ path Q of length $m-1$. We assume that the horizontal move to the right in Q is of the type So $v_{i+m-2,m-1}$, $v_{i+m-1,m-1}$. So $v_{i,1}, v_{i+1,2}$ is an upward diagonal move to the right in Q. This implies that $v_{i+1,1}$ $v_{i+2,2}$ is not an edge of G. So the horizontal move in a $v_{i+1,1} - v_{i+m,m-1}$ path of length $m-1$ in G is $v_{i+1,1}, v_{i+2,1}$ which forces the edge $v_{i+2,1}, v_{i+3,2}$ to belong to G. Continuing in this manner we see that $v_{i+2\ell,1}, v_{i+2\ell+1,2}$ is an edge for $0 \leq \ell \leq m-1$. But $v_{i+2(m-1),1} = v_{i-1,1}$. So both $v_{i-1,1}$ $v_{i,2}$ and $v_{i,1}$ $v_{i+1,2}$ are edges of G which we have already observed is not possible. Similarly we can show that if a $v_{i,1} - v_{i+m-1,m-1}$ path of length $m-1$ contains a horizontal move of the type $v_{i,1}$, $v_{i+1,1}$, then both $v_{i-2,m-2}$ $v_{i-1,m-1}$ and $v_{i-1,m-2}$ $v_{i,m-1}$ are edges of G which is impossible. Therefore $k \leq 2m-3$. ∎

To summarize the results of this section we have shown that if G is a planar $(k, 0, \{m\})$-stable graph with diam $G \geq m$, then $k \leq 2m-1$ for $m = 2$, $k \leq 2m-2$ for $m = 3$ and $k \leq 2m-3$ for $m \geq 4$ and these bounds for k are best possible.

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