## On Symmetric Designs
## Whose Incidence Matrices Satisfy a Certain Condition

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**Abstract.** Under some assumptions on the incidence matrices of symmetric designs we prove a non-existence theorem for symmetric designs. The approach generalizes Wilbrink's result on difference sets [7].

### Introduction.

A symmetric $(v, k, \lambda)$-design is an incidence structure with $v$ points and $v$ blocks of size $k$ such that any two distinct points are joined by exactly $\lambda$ blocks. The (0,1)-incidence matrix $N$ of the design is normal and satisfies

$$NN^T = N^TN = nE + \lambda J$$

where $n = k - \lambda$, $E$ denotes the $(v \times v)$-identity matrix and $J$ the $(v \times v)$-matrix whose entries are all 1, see for instance [3]. Here $N^T$ denotes the transpose of $N$. One of the main problems in combinatorics is to find necessary and sufficient conditions for the existence of such designs. It is the aim of this paper to give a necessary condition that the parameters of the design have to fulfil (under some assumptions on the matrix $N$).

The design has an automorphism group $G$ acting sharply transitively on points and blocks if and only if it has a representation as a development of a difference set. A $(v, k, \lambda)$-difference set $D$ in $G$ is a $k$-subset of $G$ such that the list of differences $d-d'$ with elements from $D$ covers every non-identity group element exactly $\lambda$ times. We label the rows and columns of a $(v \times v)$-matrix with the elements of $G$ (in the same order) and define the $(g, h)$ - entry to be 1 if $gh^{-1} \in D$ and 0 otherwise. The resulting matrix is the incidence matrix of a symmetric $(v, k, \lambda)$-design admitting $G$ as a sharply transitive automorphism group. This incidence matrix is the group invariant incidence matrix corresponding to the difference set.

A difference set in an abelian group admits $p$ as a multiplier if the map

$$g \rightarrow g^p$$

is a group automorphism and (applied to the rows and columns of the group invariant incidence matrix $N$) leaves $N$ invariant. It is not too difficult to see that

$p$ is a multiplier fixing the difference set $D$ if and only if the following matrix equation holds in GF($p$):

$$N^p = N$$

(one simply has to observe that the algebra of group invariant matrices over GF($p$) is isomorphic to the group algebra GF($p$)$G$). For background from design theory, in particular on difference sets, we refer the reader to [3]. In this short note we prove the following Theorem.

**Theorem.** *Let $N$ be the incidence matrix of a symmetric $(v, k, \lambda)$-design and assume $N^p = N$ holds in GF(p) for some prime p not dividing $v$ and not dividing $k$. If $p|(k - \lambda)$, but $p^2$ is not a divisor of $k - \lambda$, then*

$$N^{p-1} + (N^T)^{p-1} = E + v^{-1} J \in \mathrm{GF}(p)^{(v,v)}.$$

The condition $N^p = N$ is satisfied for group invariant incidence matrices corresponding to difference sets with multiplier $p$. Then the identity of our theorem reduces to the identity proved by Wilbrink and generalized by Arasu, see [1, 8]. This is the reason why our non-existence results derived from this theorem are similar to the non-existence results for difference sets contained in the just mentioned papers. However the considerable progress is that we don't make use of a sharply transitive automorphism group.

**Proof of the Theorem**

In the proof we use the following well known facts. Proofs can be found in most textbooks on Algebra.

**Results.**

(1) A matrix $N$ is diagonalizable over a field $K$ if and only if its minimum polynomial has distinct roots in $K$.
(2) A family of diagonalizable matrices can be diagonalized simultaneously if and only if the matrices commute.
(3) The polynomial $x^p - x$ has $p$ distinct roots in GF($p$).

Now we consider $N$, $N^T$, $E$ and $J$ as matrices with entries from GF($p$). Since $N^p = N$ the minimum polynomial of $N$ is a divisor of $x^p - x$ and since $N$, $N^T$ and $J$ are commuting matrices they can be diagonalized simultaneously over GF($p$). The eigenvalues of $N$ and $N^T$ are 0 or $(p-1)$-th roots of unity which shows that $N^{p-1}$ and $(N^T)^{p-1}$ have eigenvalues 0 and 1. The rank of $NN^T$ is 1 (note that $p$ divides $n$), hence the only common eigenvector of $N$ and $N^T$ corresponding to a non-zero eigenvalue is $(1 \dots 1)^T$. Since the rank of $N$ and $N^T$ is $(v+1)/2$ (see [4]) the matrix $N^{p-1} + (N^T)^{p-1}$ has eigenvalue 2 with multiplicity 1 (eigenvector is $(1 \dots 1)^T$) and eigenvalue 1 with multiplicity $v - 1$. The only element in the algebra generated by $N$, $N^T$ and $J$ with these eigenvalues is $E + v^{-1} J$. ∎

## Remarks and Applications

1. We can use our theorem to prove

**Corollary A.** *Assume $N$ is the incidence matrix of a symmetric $(v, k, \lambda)$-design of order* $n \equiv 2 \mod 4$ *satisfying* $N^2 = N$ *in* GF(2). *If $k$ is odd, then* $v = 4n-1$, $k = 2n-1$ *and* $\lambda = n-1$.

**Corollary B.** *A projective plane of order* $n \equiv 3$ *or* $6 \mod 9$ *admitting an incidence matrix $N$ with the property* $N^3 = N$ *in* GF(3) *satisfies* $n = 3$.

We omit the proofs since they are very similar to the proofs given in [7] and [1].

2. As already mentioned the condition $N^p = N$ arises naturally if $N$ is invariant under a sharply transitive automorphism group $G$ that admits $p$ as a multiplier. The author does not know about another class of symmetric designs satisfying $N^p = N$. Note that each design has many distinct incidence matrices so that it doesn't seem unlikely to find classes of designs admitting a matrix satisfying $N^p = N$.

3. The proof of the theorem is similar to the proof of Wilbrink's identity that the author gave in [6]. Here we use some knowledge about the eigenvalues of the incidence matrix $N$, in [6] we used the character values of the difference set $D$. The character values are just the eigenvalues of the group invariant incidence matrix.

4. The pattern of this note can be also used to generalize further theorems about difference sets to the more general setting of symmetric designs. This is in particular useful if the theorem about difference sets is multiplier-free, as for instance the theorem due to Mann [5]. A systematic investigation of this topic is prepared by the author.

5. If p divides $k$ we simply replace the design by its complement. The blocksize of the complement is not divisible by $p$, thus we can apply our theorem. So we also obtain a non-existence result for the case that $k$ is divisible by $p$, see [2].

## References


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