A Turán Problem for Cartesian Products of Hypergraphs

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Abstract. Let S and T be sets with $|S| = m$ and $|T| = n$. Let S_3, S_2, T_3, T_2 be the set of all 3-subsets (2-subsets) of S and T respectively. Define $Q((m,2,3),(n,2,3))$ as the smallest subset of $S_2 \times T_2$ needed to cover all elements of $S_3 \times T_3$. A more general version of this problem is initially defined, but the bulk of the investigation is devoted to studying this number. Its property as a lower bound for a planar crossing number is the reason for this focus.

1. Introduction.

We propose a generalization of Turán's problem on the Cartesian products (in the set theoretic sense) of hypergraphs. We prove an elementary bound, and then make some further observations. We conclude by showing a connection of this problem to the determination of certain planar crossing numbers.

Even in relatively simple cases we obtain constructions and results which have no direct analogue in the theory of Turán numbers. Thus, the investigation has a certain intrinsic interest, apart from possible applications.

Let n, l, and k be natural numbers with $n \geq l \geq k$. Define the function $T(n,k,l)$ as the minimum number of k-subsets of a set of size n needed to ensure every l-subset contains at least one of the k-subsets. This is the *Turán number* $T(n,k,l)$. Turán settled the case $k = 2$ (see [2] Chap. VI), but the general problem remains an intriguing open question (cf. [4]). We now present the following generalization.

Let $(n_i, k_i, l_i), i = 1, \ldots, m$, be m triples of natural numbers. Let S_i be a set where $|S_i| = n_i$ and let $S = \mathcal{P}(S_1) \times \mathcal{P}(S_2) \times \ldots \times \mathcal{P}(S_m)$. We say C is a $(n_i, k_i, l_i)_{i=1}^{n}$ *cover* if:

(1) $C \subseteq S$;
(2) whenever $x = (x_i)_{i=1}^{m} \in C$, then $|x_i| = k_i$; and
(3) for any $y = (y_i)_{i=1}^{m} \in S$ where $|y_i| = l_i$, there exists an $x = (x_i)_{i=1}^{m}$ in C with $x_i \subseteq y_i$, for all i.

Now define $Q((n_i, k_i, l_i)_{i=1}^{m})$ to be the minimum $|C|$ as C ranges over all (n_i, k_i, l_i) covers. If n_i, k_i, and l_i are independent of i, this will be referred to as $Q(n, m, k, l)$. Please note that the case $k_i = 1, i = 1$ to m, is equivalent to the problem of Zarankiewicz ([2] Chap. VII.2).

The determination of $Q(n, m, k, l)$ appears to be even more difficult than $T(n, l, k)$. This paper is largely devoted to a study of $Q(n, 2, 2, 3)$.

Note: When the parameters are understood, we will sometimes speak of C as covering S, or even covering $S_1 \times \ldots \times S_m$. This should cause no confusion.

We remark that other problems on the Cartesian product of hypergraphs have been studied, for example the chromatic number in [1].

2. Simple bounds.

The following proposition gives simple bounds on Q. $Sym(m)$ denotes the symmetric group on m elements.

Proposition 1. *Let* $\{(n_i, k_i, l_i)\}_{i=1}^{m}$ *be given. Then:*

$$\max_{\sigma \in Sym(m)} \lceil \widehat{P}(\sigma(1)) \lceil \widehat{P}(\sigma(2)) \lceil \ldots \lceil \widehat{P}(\sigma(m-1)) T(n_{\sigma(m)}, k_{\sigma(m)}, l_{\sigma(m)}) \rceil \rceil \rceil \ldots \rceil$$

$$\leq Q((n_i, k_i, l_i)_{i=1}^{m}) \leq \prod_{i=1}^{m} T(n_i, k_i, l_i).$$

where $\widehat{P}(j) = \binom{n_j}{k_j} \Big/ \binom{l_j}{k_j}, j = 1 \ldots m.$

Proof: The upper bound is obvious. For the lower bound, first set $\sigma = 1$. Now fix an l_1-subset A of S, using the notation of the previous section, and let C be an $(n_i, k_i, l_i)_{i=1}^{m}$ cover of S. Let:

$$C_A = \{x = (x_i)_{i=1}^{m} \in C : x_1 \subseteq A\}.$$

Then clearly $|C_A| \geq Q((n_i, k_i, l_i)_{i=2}^{m})$. Now $C = \bigcup_{A \subseteq S} C_A$, and any k-subset of S_1 is contained in precisely $\binom{n_1 - k_1}{n_1 - l_1}$ subsets of S_1. Thus,

$$|C| = \binom{n_1 - k}{l_1 - k_1}^{-1} \sum_{A \subseteq C_1} |C_A| \geq \binom{n_1 - k_1}{l_1 - k_1}^{-1} \binom{n_1}{l_1} Q((n_i, k_i, l_i)_{i=2}^{m})$$

$$= \widehat{P}(1) Q((n_i, k_i, l_i)_{i=2}^{m}).$$

By integrality, $|C| \geq \left\lceil \widehat{P}(1)Q((n_i,k_i,l_i)_{i=2}^{m}) \right\rceil$. Proceeding in this fashion and observing that $Q(n_m,k_m,l_m) = T(n_m,k_m,l_m)$, we obtain:

$$|C| \geq \left\lceil \widehat{P}(1) \left\lceil \cdots \left\lceil \widehat{P}(m-1)T(n_m,k_m,l_m) \right\rceil \right\rceil \cdots \right\rceil .$$

This argument is clearly independent of the order in which the indices are suppressed and C. Hence, taking the maximum over all σ in $Sym(m)$ and the min over all covers C gives the bound. ∎

Corollary 2. $\binom{n}{k}/\binom{l}{k}^{m-1}T(n,k,l) \leq Q(n,m,k,l) = T(n,k,l)^m$.

Corollary 3. *Let* $q(n) = Q(n,2,2,3)$. *Then:* $\left\lceil \frac{n(n-1)}{6} \left\lceil \frac{n(n-2)}{4} \right\rceil \right\rceil \leq q(n) \leq \left\lceil \frac{n(n-2)}{4} \right\rceil^2$

Proof: The only additional piece of information needed is that $T(n,2,3) = \left\lceil \frac{n(n-2)}{4} \right\rceil$. For this, see [2]. ∎

3. Observations on $Q(n,2,2,3)$.

For the remainder of the paper let $q(n) = Q(n,2,2,3)$. This is the simplest, non-trivial case to consider after Zarankiewicz's problem; in the next section it will be shown to have a connection with planar crossing numbers.

First, we observe that *Corollary 3* yields the following bounds:

$4 \leq q(4) \leq 4$, $14 \leq q(5) \leq 16$, $30 \leq q(6) \leq 36$, $63 \leq q(7) \leq 81$, and $112 \leq q(8) \leq 144$.

Thus, $q(4) = 4$. The next sequence of propositions gives the exact values for these small n.

Proposition 4. $q(5) = 14$

Proof: Let $S = \{1,2,3,4,5\}$ and $T = \{1',2',3',4',5'\}$, then the 14 pairs of doubletons in Table I show that $q(5) \leq 14$ and hence by Corollary 3 $q(5) = 14$. ∎

Table I

$(\{1,2\},\{1',2'\})$	$(\{1,2\},\{5',3'\})$	$(\{1,5\},\{5',4'\})$	$(\{2,5\},\{3',4'\})$
$(\{1,4\},\{2',4'\})$	$(\{2,4\},\{1',4'\})$	$(\{1,3\},\{3',4'\})$	$(\{2,3\},\{4',5'\})$
$(\{4,5\},\{2',5'\})$	$(\{3,4\},\{1',5'\})$	$(\{3,4\},\{2',3'\})$	$(\{3,5\},\{3',5'\})$
$(\{3,5\},\{2',4'\})$	$(\{4,5\},\{1',3'\})$		

Proposition 5. $q(6) = 32$

Proof: Let $S = \{1,2,3,4,5,6\}$ and $T = \{1',2',3',4',5',6'\}$, then the 32 pairs of doubletons in Table II show that $q(6) \leq 32$. ∎

Table II

$(\{1,2\},\{1',2'\})$	$(\{1,2\},\{4',5'\})$	$(\{1,3\},\{1',3'\})$	$(\{1,3\},\{5',6'\})$
$(\{2,3\},\{2',3'\})$	$(\{2,3\},\{4',6'\})$	$(\{2,4\},\{3',5'\})$	$(\{2,4\},\{1',6'\})$
$(\{1,4\},\{3',4'\})$	$(\{1,4\},\{2',6'\})$	$(\{3,4\},\{1',4'\})$	$(\{3,4\},\{2',5'\})$
$(\{3,5\},\{1',2'\})$	$(\{3,5\},\{4',5'\})$	$(\{1,5\},\{4',6'\})$	$(\{1,5\},\{2',3'\})$
$(\{2,5\},\{1',3'\})$	$(\{2,5\},\{5',6'\})$	$(\{4,5\},\{1',5'\})$	$(\{4,5\},\{2',4'\})$
$(\{4,5\},\{3',6'\})$	$(\{1,6\},\{3',6'\})$	$(\{1,6\},\{2',4'\})$	$(\{1,6\},\{1',5'\})$
$(\{2,6\},\{2',5'\})$	$(\{2,6\},\{1',4'\})$	$(\{4,6\},\{4',6'\})$	$(\{4,6\},\{2',3'\})$
$(\{3,6\},\{1',6'\})$	$(\{3,6\},\{3',5'\})$	$(\{5,6\},\{2',6'\})$	$(\{5,6\},\{3',4'\})$

To show that $q(6) \geq 32$ we require the following lemma:

Lemma 6.

(a) $Q((4,2,3),(5,2,3)) = 8$
(b) $Q((4,2,3),(6,2,3)) = 12$
(c) $Q((5,2,3),(6,2,3)) = 21$.

Proof: (a) and (b) follow from Proposition 1. To prove (c), let $|S_1| = 5$ and $|S_2| = 6$, and let C be a $((5,2,3),(6,2,3))$ cover of size 20. Proceeding as in Proposition 1, this requires every triangle in S_2 to be covered exactly four times by C. Let x be the number of times $\{1',2'\}$ appears as the second component in C (here $S_2 = \{1',2',3',4',5',6'\}$. Now C acts as a $((5,2,3),(4,2,3))$ cover on $S_1 \times \{3,4,5,6\}$. By part (a) of the lemma and the fact every triangle of S_2 is covered four times by C, we see C restricted to $S_1 \times \{3,4,5,6\}$ has size 8. The four triangles containing $\{1',2'\}$ contribute $16 - 3x$ more elements to C, so $|C| = 24 - 3x$. But as x is integral $|C| \neq 20$. Hence, $|C| \geq 21$. Now observing that the $((6,2,3),(6,2,3))$ cover given previously acts as a $((5,2,3),(6,2,3))$ cover of size 21 when restricted to $\{1,2,3,4,5\} \times S_2$ proves the lemma. ∎

We now return to the proof of Proposition 15. Let C be a $((6,2,3),(6,2,3))$ cover; let S_1 and S_2 be the respective sets of size 6; and suppose $|C| \leq 32$. Let x be the number of times $\{1,2\}$ appears as the first component of C. Now C acts as a $((4,2,3),(6,2,3))$ cover on $\{3,4,5,6\} \times S_2$. Hence the triangle from $\{3,4,5,6\}$ contributes at least 12 elements to C (Lemma 5,(b)). The triangles containing $\{1',2'\}$ contribute at least $24 - 3x$ more elements, hence, $|C| \geq 36 - 3x$. Thus $x \geq 2$. By identical reasoning, every edge of S_1 appears at least twice as the first component in C.

Now C acts as $((5,2,3),(6,2,3))$ cover on $\{1,2,3,4,5\} \times S_1$, and hence has size at least 21 after restriction (Lemma 6, (c)). Thus, at least one edge of

$\{1,2,3,4,5\}$ occurs at least three times as a first component in C. Let this edge be (12). Now C also acts as a $((5,2,3),(6,2,3))$ cover on $\{2,3,4,5,6\}\times S_2$. Thus, one edge of $\{2,3,4,5,6\}$ must occur at least three times in C. But now two edges in S_1 contribute at least three times to C, while all other edges contribute at least twice to C. Hence, $|C| \geq 32$, and $q(6) = 32$. ∎

Proposition 7. *$q(7) = 63$ and $q(8) = 112$.*

Proof: We first show that $q(8) = 112$. Let $S = \{1,2,3,\ldots,8\}, T = \{1',2',3',\ldots,8'\}$ and let $\mathcal{B}$ be a Steiner quadruple system of order 8 on S. Define C by

$$C = \{(\{a,b\},\{c',d'\}) : \{a,b,c,d\} \in \mathcal{B}\} \bigcup \{(\{a,b\},\{a',b'\}) : a,b \in S\}.$$

Recall that in a Steiner quadruple system of order 8 every pair $P = \{a,b\}$ is in exactly three quadruples, say B_1, B_2, B_3, and that the pairs $P, B_1 - P, B_2 - P, B_3 - P$, are a partition of S. Thus $|C| = 112$ as desired. Moreover, given any triple $\{x,y,z\} \subseteq S$ it is easy to see that the graph on T induced by the set of pairs $\{c',d'\}$ such that $(\{a,b\},\{c',d'\}) \in C$ and $a,b \in \{x,y,z\}$ is a disjoint union of two K_4's. Thus every pair of triples one from S and one from T must contain one of the pairs of doubletons in C. Whence $q(8) \leq 112$ and so $q(8) = 112$ by Corollary 3.

To show $q(7) = 63$ take C' to be the 63 pairs of doubletons in C that do not contain 8 or $8'$. ∎

The result for $q(6)$ seems anomalous because of its failure to meet the lower bound specified by Corollary 2. The next proposition shows that it is really 5, 7, and 8 that are the exceptional cases.

Proposition 8. *Let $n \geq 9$. Then*

$$q(n) > \left\lceil \frac{n(n-1)}{6} \left\lceil \frac{n(n-2)}{4} \right\rceil \right\rceil.$$

Proof: Recall that Corollary 3 is derived by observing that $\binom{n}{3} \lceil \frac{n(n-2)}{4} \rceil \leq q(n) \cdot (n-2)$. It follows from the proof of Corollary 3 that every triangle in S_1 contains an edge associated with at least $\frac{1}{3} \lceil \frac{n(n-2)}{4} \rceil$ edges in any minimal cover. Now if $n \geq 9, : \frac{1}{3} \lceil \frac{n(n-2)}{4} \rceil > \frac{n}{2}$. So at least one triangle in S_2 is covered twice by an edge in S_1 which occurs at least $\frac{1}{3} \lceil \frac{n(n-2)}{4} \rceil$ times in the minimal cover. Let x be the number of edges in S_1 with this property.

Now "reverse the orientation" to establish the lower bound. Each triangle in S_2 must be covered at least $\lceil \frac{n(n-2)}{4} \rceil$ times. However, every triangle covered twice by the same edge must be covered at least $\lceil \frac{n(n-2)}{4} \rceil + 1$ times. Hence: $\binom{n}{3} \lceil \frac{n(n-2)}{4} \rceil + x \leq q(n)(n-2)$ Since every triangle in S_1 contributes at least one edge with the above property, $x \geq \lceil \frac{n(n-2)}{4} \rceil$. Thus: $\frac{n(n-1)}{6} \lceil \frac{n(n-2)}{4} \rceil + \frac{\lceil \frac{n(n-2)}{4} \rceil}{(n-2)} \leq$ $q(n)$ and so $q(n) > \lceil \frac{n(n-1)}{6} \lceil \frac{n(n-2)}{4} \rceil \rceil$ for all $n \geq 9$ ∎

We now discuss an asymptotic lower bound on $Q(m,n) = Q((m,2,3),(n,2,3))$. An easy counting argument, left to the reader, shows that the ratio $Q(m,n) / \binom{m}{2}\binom{n}{2}$ is monotone non-decreasing in each of the arguments m and n. Thus, since $q(n) = Q(n,n)$ the limit

$$q = \lim_{n \to \infty} \frac{q(n)}{\binom{n}{2}^2}$$

exists; further the bounds in Corollary 3 imply that $1/6 \leq q \leq 1/4$. (It is appealing to think of q as the "asymptotic proportion" of "2×2" sets needed to cover all "3×3" sets.)

Theorem 9. $q \geq \frac{3-\sqrt{5}}{4} \approx 0.1910 > \frac{1}{6} \approx 0.1667$.

Before proceeding with the proof, we need some notation and a lemma. Let X and Y be disjoint n-element sets and let C be a fixed system of (2×2)-sets $p_1 \times p_2$ ($p_1 \subseteq X, |p_1| = 2, p_2 \subseteq Y, |p_2| = 2$) that cover all the (3×3)-sets. Each pair p in X then has an associated graph $G_Y(p) = \{q \subseteq Y : p \times q \in C\}$; symmetrically each p in Y has an associated graph $G_X(p)$. Note $|C| = \sum_{p \subseteq X} |G_Y(p)| = \sum_{p \subseteq Y} |G_X(p)|$. If T is any 3-element subset of X, then

$$\left| \bigcup_{p \in \binom{T}{2}} G_Y(p) \right| \geq T(n,3,2) = \left\lceil \frac{n(n-2)}{4} \right\rceil \tag{1}$$

since the graph $G_Y(T) = \bigcup_{p \in \binom{T}{2}} G_Y(p)$ has the property that every 3-element subset of Y contains an edge of $G_Y(T)$.

Lemma 10 **(Refinement of Boole's inequality).** *Let $B_1, B_2, \ldots, B_m$ be any sequence of finite sets. Then*

$$|B_1 \cup B_2 \cup \ldots \cup B_m| \leq \sum_{i=1}^{m} |B_i| - \frac{2}{\Delta} \sum_{1 \leq i < j \leq m} |B_i \cap B_j| \tag{2}$$

where $\Delta = \max_x[\deg(x) = |\{i : x \in B_i\}|]$ is the maximum degree of points.

Proof: Note that, putting $V = B_1 \cup B_2 \cup \ldots \cup B_m$,

$$0 \leq \sum_{x \in V} (\deg(x) - 1)(\Delta - \deg(x)) \tag{3}$$

because each term of this sum is non-negative. Expanding and using the identities

$$\sum_{x\in V} \deg(x) = \sum_{i=1}^{m} |B_i| \tag{4}$$

$$\sum_{x\in V} \binom{\deg(x)}{2} = \sum_{1\le i<j\le m} |B_i \cap B_j| \tag{5}$$

one easily obtains (2). ∎

We apply this lemma in the case $m = 3$ only. If the triplet T contains the pairs p_1, p_2, p_3, then Lemma 10 applied to (1) gives

$$\left\lceil \frac{n(n-2)}{4} \right\rceil \le \sum_{i=1}^{3} |G_Y(p_i)| - \frac{2}{3} \sum_{i<j} |G_Y(p_i) \cup G_Y(p_j)| \tag{6}$$

Summing (6) over all triplests $T \subset X$, we get

$$\binom{n}{3} \left\lceil \frac{n(n-2)}{4} \right\rceil \le \sum_{T} \sum_{i=1}^{3} |G_Y(p_i)| - \frac{2}{3} \sum_{T} \sum_{i<j} |G_Y(p_i) \cap G_Y(p_j)| \tag{7}$$

$$= (n-2) \sum_{p\in\binom{X}{2}} |G_Y(p)| - \frac{2}{3} \sum_{q\in\binom{Y}{2}} \sum_{v\in X} \binom{\deg(G_X(q), v)}{2} \tag{8}$$

where $\deg(G_X(q), v)$ is the degree of vertex v in the graph $G_X(q)$. (To see the equality of the last terms of (7) and (8), observe that both sums count the total number of unorderd pairs of incident edges, scanning over all graphs $G_X(q)$.) Now by Jensen's inequality on convex functions,

$$\begin{aligned} \sum_{v\in X} \binom{\deg(G_X(q), v)}{2} &\ge n \binom{\frac{1}{n}\sum_{v\in X} \deg(G_X(q), v)}{2} \\ &= n \binom{2|G_X(q)|/n}{2} \\ &= |G_X(q)| \left(2|G_X(q)|/n - 1\right) \end{aligned} \tag{9}$$

and hence

$$
\begin{aligned}
\sum_{q\in\binom{Y}{2}}\sum_{v\in X}\binom{\deg(G_X(q),v)}{2} &\geq \sum_{q\in\binom{Y}{2}} |G_X(q)|\,(2|G_X(q)|/n-1) \\
&= \frac{2}{n}\sum_{q\in\binom{Y}{2}} |G_X(q)|^2 - \sum_{q\in\binom{Y}{2}} |G_X(q)| \\
&\geq \frac{2}{n}\frac{1}{\binom{n}{2}}\left[\sum_{q\in\binom{Y}{2}} |G_X(q)|\right]^2 - \sum_{q\in\binom{Y}{2}} |G_X(q)| \\
&= \frac{4}{n^2(n-1)}|C|^2 - |C|.
\end{aligned} \tag{10}
$$

Substituting (10) into (8) gives

$$
\begin{aligned}
\binom{n}{3}\left\lceil\frac{n(n-2)}{4}\right\rceil &\leq (n-2)|C| - \frac{8}{3n^2(n-1)}|C|^2 + \frac{2}{3}|C| \\
&= \left(n-\frac{4}{3}\right)|C| - \frac{8}{3n^2(n-1)}|C|^2.
\end{aligned} \tag{11}
$$

We may assume that $|C| = q(n) = c_n\binom{n}{2}^2$ with $\lim_{n\to\infty} c_n = q$. Hence dividing both sides of (11) by $n\binom{n}{2}^2$ and letting n tend to infinity, we get

$$q^2 - \frac{3}{2}q + \frac{1}{4} \leq 0$$

and so

$$q \geq \frac{1}{2}\left\{\frac{3}{2} - \sqrt{\frac{9}{4} - 1}\right\} = \frac{3-\sqrt{5}}{4}$$

which proves Theorem 9.

4. Crossing numbers.

The *crossing number* $\nu(G)$ of a graph G is the minimum number of crossings (of its edges) among the drawings of G in the plane. The connection between $\nu(K_{m,n})$ and and the preceding theory is given by:

Proposition 11. *For all* $m, n \geq 3$: $Q((m,2,3),(n,2,3)) \leq \nu(K_{m,n}) \leq T(m,2,3)\cdot T(n,2,3)$.

Proof: The upper bound on $\nu(K_{m,n})$ was first established by Zarankiewicz [6] who erroneously thought he had proved equality. For the lower bound let $f: G \to$

$\Re^2$ be some planar embedding of G. Now it is well known that $\nu(3,3) = 1$. Hence, given any triples $\{a,b,c\} \subseteq S$ and $\{d,e,f\} \subseteq T$, there must exist at least one pair of doubletons (say $\{a,b\},\{d,e\}$) which generate a cross in $f(G)$. If each cross in $f(G)$ is identified with the pair of doubletons from S and T which generate it, we see each pair of triples must be covered by at least one such pair of doubletons. Thus, $Q((m,2,3,),(n,2,3)) \leq \nu(K_{m,n})$. ∎

As an application, we have easily:

Corollary 12. *For all* $n \geq 3, \nu(4,n) = 2 \cdot \lceil \frac{n(n-2)}{4} \rceil$

Proof: From Proposition 1, $Q((4,2,3),(n,2,3)) = T(4,2,3) \cdot T(n,2,3)$ $= 2 \cdot \lceil \frac{n(n-2)}{4} \rceil$. ∎

The best general result on the crossing numbers of complete bipartite graphs is the following due to Kleitman [5].

For $1 \leq \min\{m,n\} \leq 6$,

$$\nu(K_{m,n}) = \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor.$$

The smallest complete bipartite graph whose crossing number is unknown is $K_{7,7}$. In [5] it is shown that $\nu(K_{7,7}) \in \{77, 79, 81\}$.

A similar connection between $\nu(K_n)$ and $T(n,5,4)$, due to Ringel is discussed in [3].

Acknowledgements.

The first author was supported by NSERC grant U0463.

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