On the Existence of Simple and Indecomposable Block Designs with Block Size 4

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Abstract. It is proved in this paper that for $\lambda = 4$ and 5, the necessary conditions for the existence of a simple $B(4,\lambda;v)$ are also sufficient. It is also proved that for $\lambda = 4$ and 5, the necessary conditions for the existence of an indecomposable simple $B(4,\lambda;v)$ are also sufficient, with the unique exception $(v,\lambda) = (7,4)$ and 10 possible exceptions.

1. Introduction.

A balanced incomplete block design $B(k,\lambda;v)$ is an ordered pair $(V,\mathcal{B})$ where V is a finite set containing v points, and $\mathcal{B}$ is a collection of k-subsets (called blocks) of V such that each pair of distinct points of V is contained in exactly λ blocks. A $B(k,\lambda;v)$ is called simple and denoted $NB(k,\lambda;v)$ if it contains no repeated blocks.

Let $(V,\mathcal{B})$ be a $B(k,\lambda;v)$, if there exist $\mathcal{B}_1 \subset \mathcal{B}$ and $1 \leq \lambda_1 < \lambda$ such that $(V,\mathcal{B}_1)$ is a $B(k,\lambda_1;v)$ then $(V,\mathcal{B})$ is called decomposable. Otherwise it is called indecomposable.

It is not difficult to verify that the following conditions are necessary for the existence of an $NB(k,\lambda;v)$ or an indecomposable $NB(k,\lambda;v)$:

$$\begin{aligned} \lambda(v-1) &\equiv 0 \pmod{(k-1)}, \\ \lambda v(v-1) &\equiv 0 \pmod{(k(k-1))}, \\ \lambda &\leq \binom{v-2}{k-2}. \end{aligned} \tag{1}$$

For given k and λ, any positive integer v satisfying (1) is called admissible.

A $B(k,1;v)$ is also known as a Steiner system. Obviously any Steiner system is both simple and indecomposable.

For given k and $\lambda \geq 2$, the existence of $NB(k,\lambda;v)$'s has been studied by several authors. It was proved that there exists an $NB(3,2;v)$ if and only if $v \equiv 0$ or $1 \pmod 3$, $v \geq 4$ [1] and there exists an $NB(3,3;v)$ if and only if $v \equiv 1 \pmod 2$, $v \geq 5$ [2]. The existence of simple triple systems for arbitrary λ was completely determined in [4]: there exists an $NB(3,\lambda;v)$ for every admissible v.

Much less is known concerning the existence of indecomposable $NB(k,\lambda;v)$'s. To the author's knowledge, the problem is completely solved only in the following cases:

(1) [7] There exists an indecomposable $NB(3,2;v)$ if and only if

$$v \equiv 0 \text{ or } 1 \pmod 3, \quad v > 3 \text{ and } v \neq 7;$$

there exists an indecomposable $NB(3,3;v)$ if and only if

$$v \equiv 1 \pmod 2, \quad v \geq 5.$$

(2) [3] There exists an indecomposable $NB(3,4;v)$ if and only if

$$v \equiv 0 \text{ or } 1 \pmod 3, \quad v \geq 10.$$

(3) [8] There exists an indecomposable $NB(4,2;v)$ if and only if

$$v \equiv 1 \pmod 3, \quad v \geq 7.$$

(4) [6] There exists an indecomposable $NB(4,3;v)$ if and only if

$$v \equiv 0 \text{ or } 1 \pmod 4, \quad v \geq 5.$$

In this paper we will discuss the existence of $NB(4,\lambda;v)$ and indecomposable $NB(4,\lambda;v)$, and give a complete solution to the existence of $NB(4,4;v)$ and $NB(4,5;v)$. Further, we will also give an almost complete solution to the existence of indecomposable $NB(4,4;v)$ and indecomposable $NB(4,5;v)$.

2. Simple designs containing a given subdesign.

A pairwise balanced design $B(K,\lambda;v)$ is an ordered pair $(V,\mathcal{A})$ where V is a v-set and $\mathcal{A}$ is a collection of subsets (called blocks) of V such that $|B| \in K$ for each block $B \in \mathcal{A}$, and each pair of distinct points of V is contained in exactly λ blocks. If $K = \{k\}$, then a $B(\{k\},\lambda;v)$ is in fact a $B(k,\lambda;v)$.

Let $(V,\mathcal{A})$ be a $B(K,\lambda;v)$. Let $V_1 \subset V$, $|V_1| = v_1$, $\mathcal{A}_1 \subset \mathcal{A}$. If $(V,\mathcal{A}_1)$ is a $B(K,\lambda;v_1)$ then it is called a subdesign of $(V,\mathcal{A})$, or it is embedded in $(V,\mathcal{A})$.

The following lemma will be used frequently in this paper and the proof is obvious:

Lemma 1. *Let $(V,\mathcal{A})$ be a $B(k,\lambda;v)$ and $(V_1,\mathcal{A}_1)$ be a subdesign of $(V,\mathcal{A})$. If $(V_1,\mathcal{A}_1)$ is indecomposable, then $(V,\mathcal{A})$ is also indecomposable.*

We will also need the concept of incomplete pairwise balanced designs. An incomplete pairwise balanced design $(v,w;K,\lambda)$-IPBD is an ordered triple $(X,Y,\mathcal{A})$ where X is a v-set, Y is a w-subset of X and $\mathcal{A}$ is a collection of subsets (called blocks) of X such that $|B| \in K$ and $|B \cap Y| \leq 1$ for each $B \in \mathcal{A}$, and each pair of distinct points of X, not both in Y, is contained in exactly λ blocks.

Lemma 2. *Let* $(X,Y,\mathcal{A})$ *be a* $(v,w;\{k\};\lambda)$*-IPBD and let*

$$\mathcal{A}_0=\{B\in\mathcal{A}\mid |B\cap Y|=0\},\quad \mathcal{A}_1=\{B\in\mathcal{A}\mid |B\cap Y|=1\}.$$

Then

$$|\mathcal{A}_0|=\lambda(v-w)\{v-(k-1)w-1\}/k(k-1),\ |\mathcal{A}_1|=\lambda w(v-w)/(k-1).\quad (2)$$

Proof: For any $y\in Y$ and $x\in X\backslash Y$, $\{x,y\}$ is contained in exactly λ blocks. So for any $y\in Y$, y is contained in precisely $\lambda(v-w)/(k-1)$ blocks. As $|Y|=w$, we have $|\mathcal{A}_1|=\lambda w(v-w)/(k-1)$. Obviously,

$$|\mathcal{A}|=\lambda\left\{\binom{v}{2}-\binom{w}{2}\right\}\Big/\binom{k}{2},$$

thus, we have

$$|\mathcal{A}_0|=|\mathcal{A}|-|\mathcal{A}_1|=\lambda(v-w)\cdot\{v-(k-1)w-1\}/k(k-1).$$

∎

Theorem 1. *If there exists a simple* $(v,w;\{k\},\lambda_1)$*-IPBD, a simple* $(v,w;\{k\},\lambda_2)$*-IPBD and* $\lambda_1\lambda_2(k-2)!\,(v-w)\{kw(v-w-k+1)+(v-(k-1)w-1)^2\}(v-w-k)!<k(k-1)\cdot(v-w-1)!$*. Then there exists a simple* $(v,w;\{k\},\lambda_1+\lambda_2)$*-IPBD.*

Proof: Let $(X,Y,\mathcal{A})$ and $(X,Y,\mathcal{B})$ be a simple $(v,w;\{k\},\lambda_1)$-IPBD and a simple $(v,w;\{k\},\lambda_2)$-IPBD, respectively. Let S be the symmetric group on X and $\pi\in S$ be a permutation. For each subset $M=\{x_1,x_2,\dots,x_m\}$ of X, let

$$\pi(M)=\{\pi(x_1),\pi(x_2),\dots,\pi(x_m)\},\quad \pi(\mathcal{A})=\{\pi(B)\mid B\in\mathcal{A}\}.$$

Let G be the subgroup of S fixing Y, then $|G|=w!\,(v-w)!$ and for any $\pi\in G$, $(X,Y,\pi(\mathcal{A}))$ is also a $(v,w;\{k\},\lambda_1)$-IPBD.

Now for two given blocks $B_1\in\mathcal{A}$ and $B_2\in\mathcal{B}$, if $|B_1\cap Y|\neq|B_2\cap Y|$, then there does not exist $\pi\in G$ such that $\pi(B_1)=B_2$. If $|B_1\cap Y|=|B_2\cap Y|=0$, then the number of such permutations π is $w!k!(v-w-k)!$. If $|B_1\cap Y|=|B_2\cap Y|=1$, then the number of such permutations is $(w-1)!\,(v-w-k+1)!\,(k-1)!$.

Let n be the number of permutations $\pi\in G$ such that

$$|\pi(\mathcal{A})\cap\mathcal{B}|\geq 1,$$

then, by Lemma 2, we have

$$n \le \frac{\lambda_1\lambda_2 w^2(v-w)^2}{(k-1)^2}(w-1)!(k-1)!(v-w-k+1)!$$
$$+\frac{\lambda_1\lambda_2(v-w)^2!(v-(k-1)w-1)^2}{k^2(k-1)^2}w!k!(v-w-k)!$$
$$=\lambda_1\lambda_2(k-2)!(v-w)^2(kw(v-w-k+1)$$
$$+(v-(k-1)w-1)^2)w!(v-w-k)!/k(k-1)$$
$$< w!(v-w)!.$$

Thus there exists a permutation $\pi \in G$ such that $\pi(\mathcal{A})$ and $\mathcal{B}$ share no common blocks and therefore $(X, Y, \mathcal{B} \cup \pi(\mathcal{A}))$ is a simple $(v, w; \{k\}, \lambda_1 + \lambda_2)$-IPBD. ∎

3. Existence of $NB(4,4;v)$ and $NB(4,5;v)$.

The purpose of this section is to prove that for $\lambda = 4$ and 5, the necessary conditions (1) for the existence of an $NB(4,\lambda;v)$ are also sufficient. The following result is needed:

Lemma 3 [8]. *There exists an $NB(4,2;v)$ if and only if*

$$v \equiv 1 \pmod 3, \quad v \ge 7.$$

Theorem 2. *There exists an $NB(4,4;v)$ if and only if*

$$v \equiv 1 \pmod 3, \quad v \ge 7.$$

Proof: We note that a $(v,0;\{k\},\lambda)$-IPBD is in fact a $B(k,\lambda;v)$. Thus, by Lemma 3, there is a simple $(v,0;\{4\},2)$-IPBD for every $v \equiv 1 \pmod 3$, $v \ge 7$. Now let $w \equiv 0$ and $\lambda_1 = \lambda_2 = 2$ in Theorem 1, for each $v \equiv 1 \pmod 3$, $v \ge 13$, we obtain an $NB(4,4;v)$ from an $NB(4,2;v)$. To complete the proof of the theorem, we construct an $NB(4,4;7)$ and an $NB(4,4;10)$ as follows:

$NB(4,4;7)$: $X = Z_7$,
$\mathcal{A}$: $\{0,1,2,4\}$, $\{0,1,2,5\} \pmod 7$.

$NB(4,4;10)$: $X = Z_{10}$,
$\mathcal{A}$: $\{0,1,3,5\}$, $\{0,1,3,7\}$, $\{0,1,2,7\} \pmod{10}$.

∎

Theorem 3. *There exists an $NB(4,5;v)$ if and only if*

$$v \equiv 1 \text{ or } 4 \pmod{12}, \quad v \ge 13.$$

Proof: The condition is necessary. To proof the sufficiency, for each $v \equiv 1$ or $4 \pmod{12}$, $v \ge 13$, let $\lambda_1 = 1$, $\lambda_2 = 4$ and $w = 0$ in Theorem 1, we obtain an $NB(4,5;v)$ from an $NB(4,1;v)$ and an $NB(4,4;v)$. This completes the proof. ∎

4. Indecomposability.

In this section, we will discuss the existence of indecomposable $NB(4,\lambda;v)$ for $\lambda = 4$ and 5, and give an almost complete solution.

A transversal design $TD(k,\lambda;n)$ is an ordered triple $(X,\mathcal{G},\mathcal{A})$ where X is a v-set, $v = kn$, $\mathcal{G}$ is a collection of n-subsets (called groups) of X, $\mathcal{G}$ partitions X, and $\mathcal{A}$ is a collection of k-subsets (called blocks) such that each block intersects every group in a unique point, and each pair of points from distinct groups appears in exactly λ blocks. A $TD(k,\lambda;n)$ is called simple if it contains no repeated blocks. It is well known that the existence of a $TD(k,1;n)$ is equivalent to the existence of $k-2$ mutually orthogonal Latin squares of order n.

Lemma 4. *If there is an $NB(4,\lambda;v)$, then any $NB(4,\lambda;v)$ can be embedded in an $NB(4,\lambda;4v)$ if $v \neq 6$ and $\lambda \leq v$, or an $NB(4,\lambda;4(v-1)+1)$ if $v \neq 7$ and $\lambda \leq v-1$.*

Proof: If $v \geq 3$ and $v \neq 6$, then there exists a $TD(4,1;v)$. Let $\mathcal{G} = \{G_1, G_2, G_3, G_4\}$, $X = \cup_{i=1}^{4} G_i$ and $(X,\mathcal{G},\mathcal{A}_0)$ be a $TD(4,1;v)$, where $G_1 = Z_v, G_2, G_3$ and G_4 are four disjoint v-sets. For $1 \leq \lambda \leq v$ and $i = 0,1,\ldots,\lambda-1$, let

$$\mathcal{A}_i = \{\{a+i,b,c,d\} \mid \{a,b,c,d\} \in \mathcal{A}_0, (a,b,c,d) \in G_1 \times G_2 \times G_3 \times G_4\},$$
$$\mathcal{A} = \cup_{i=0}^{\lambda-1} \mathcal{A}_i.$$

Then $(X,\mathcal{G},\mathcal{A})$ is a simple $TD(4,\lambda;v)$.

Now for each $i = 1,2,3,4$, form an $NB(4,\lambda;v)$ on G_i and denote the block set by $\mathcal{B}_i$. Let

$$\mathcal{B} = \mathcal{A} \cup \{\cup_{i=1}^{4} \mathcal{B}_i\},$$

then $(X,\mathcal{B})$ is an $NB(4,\lambda;4v)$ which contains each $(G_i,\mathcal{B}_i)$ as a subdesign.

If $v > 4$, $v \neq 7$ and $\lambda \leq v-1$, then there is a simple $TD(4,\lambda;v-1)$. Let $(X,\mathcal{G},\mathcal{A})$ be a simple $TD(4,\lambda;v-1)$ where $\mathcal{G} = \{G_1,G_2,G_3,G_4\}$. Let ∞ be a new element, form an $NB(4,\lambda;v)$ on $G_i \cup \{\infty\}$ and denote the block set by $\mathcal{B}_i$, $i = 1,2,3,4$. Let

$$\mathcal{B} = \mathcal{A} \cup \{\cup_{i=1}^{4} \mathcal{B}_i\},$$

then $(X \cup \{\infty\},\mathcal{B})$ is an $NB(4,\lambda;4(v-1)+1)$ which contains each $(G_i,\mathcal{B}_i)$ as a subdesign. ∎

Let $(X,\mathcal{B})$ be a $B(k,\lambda;v)$, a parallel class is a subcollection $\mathcal{B}'$ of $\mathcal{B}$, $\mathcal{B}'$ partitions X. If $\mathcal{B}$ can be partitioned into parallel classes, then $(X,\mathcal{B})$ is called resolvable and a resolvable $B(k,\lambda;v)$ is denoted $RB(k,\lambda;v)$. It is well known that there exists an $RB(3,1;v)$ if and only if $v \equiv 3 \pmod 6$ [9].

Lemma 5. *If* $v \equiv 1 \pmod 3$, $\lambda \leq v$, *and there is an* $NB(4,\lambda;v)$, *then any* $NB(4,\lambda;v)$ *can be embedded in an* $NB(4,\lambda;3v+1)$.

Proof: Let $V = \{\infty_0, \infty_1, \ldots, \infty_{v-1}\}$ and $(V, \mathcal{B})$ be an $NB(4,\lambda;v)$. As $v \equiv 1 \pmod 3$, there exists an $RB(3,1;2v+1)$. Let $(X, \mathcal{A})$ be an $RB(3,1;2v+1)$ and let $\mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_{v-1}$ be the parallel classes. Let

$$\mathcal{B}_{ij} = \{\{a,b,c,\infty_{i+j}\} \mid \{a,b,c\} \in \mathcal{A}_i\}, \quad i,j \in Z_v,$$
$$\mathcal{B}_i = \cup_{j=1}^{\lambda-1} A_{ij}, \quad i \in Z_v.$$

Then $(V \cup X,\ \mathcal{B} \cup \{\cup_{i=0}^{v-1} \mathcal{B}_i\})$ is an $NB(4,\lambda;3v+1)$ which contains $(V, \mathcal{B})$ as a subdesign. ∎

The following result is useful in our construction of indecomposable $NB(4,\lambda;v)$ for $\lambda = 4$ and 5.

Lemma 6 [10]. *If* $v, w \equiv 1$ *or* $4 \pmod{12}$, *and* $v \geq 3w+1$, *or* $v, w \equiv 7$ *or* $10 \pmod{12}$ *and* $v \geq 3w+1$, *then there exists a* $(v, w; \{4\}, 1)$*-IPBD.*

Theorem 4. *There exists an indecomposable* $NB(4,4;v)$ *if and only if*

$$v \equiv 1 \pmod 3, \quad v \geq 10,$$

with the following 6 possible exceptions:

$$v = 16, 19, 22, 25, 28, 34.$$

Proof: The existence of an indecomposable $NB(4,4;7)$ is equivalent to the existence of an indecomposable $NB(3,2;7)$. It has been proved (see [5]) that there does not exist an indecomposable $NB(3,2;7)$, so there does not exist an indecomposable $NB(4,4;7)$.

We form an $NB(4,4;10)$ and an $NB(4,4;13)$ as follows:

$$NB(4,4;10): \quad X = Z_{10},$$

Base blocks: $\{0,1,2,7\}$, $\{0,1,3,7\}$, $\{0,1,3,5\}$ (mod 10).

$$NB(4,4;13): \quad X = Z_{13},$$

Base blocks: $\{0,1,3,8\}$, $\{0,1,6,10\}$, $\{0,2,6,8\}$, $\{0,1,3,4\}$ (mod 13).

It can be checked that both of the above designs are indecomposable, we omit the details here.

Now let $v = 10$ in Lemma 4, we obtain an indecomposable $NB(4,4;37)$ and an indecomposable $NB(4,4;40)$ from an indecomposable $NB(4,4;10)$. Let $v = 13$, we obtain an indecomposable $NB(4,4;49)$ and an indecomposable $NB(4,4;52)$ from an indecomposable $NB(4,4;13)$. Let $v = 10$ in Lemma 5, we obtain an indecomposable $NB(4,4;31)$.

Now let $w = 10$ in Theorem 1, then for every $v \equiv 7$ or $10 \pmod{12}$ and $v \geq 43$, there exists a simple $(v, 10; \{4\}, 4)$-IPBD. Let $(X, Y, \mathcal{A})$ be such a simple $(v, 10; \{4\}, 4)$-IPBD, form an indecomposable $NB(4,4;10)$ on Y and denote the block set by $\mathcal{B}$, then obviously $(X, \mathcal{A} \cup \mathcal{B})$ is an indecomposable $NB(4,4;v)$.

Let $v = 13$ in Theorem 1, then for every $v \equiv 1$ or $4 \pmod{12}$ and $v \geq 61$, there exists a simple $(v, 13; \{4\}, 4)$-IPBD. Let $(X, Y, \mathcal{A})$ be such a simple $(v, 13; \{4\}, 4)$-IPBD. Form an indecomposable $NB(4,4;13)$ on Y and denote the block set by $\mathcal{B}$, then $(X, \mathcal{A} \cup \mathcal{B})$ is an indecomposable $NB(4,4;v)$.

Obviously if there exists an indecomposable $NB(4,4;v)$, then $v \equiv 1 \pmod 3$ and $v \geq 10$. This completes the proof. ∎

Theorem 5. *There exists an indecomposable $NB(4,5;v)$ if and only if*

$$v \equiv 1 \text{ or } 4 \pmod{12}, \quad v \geq 13,$$

with the possible exceptions: $v = 16, 25, 28$ and 37.

Proof: We show that there exists an indecomposable $NB(4,5;13)$ by the following direct construction:

$$X = Z_{13},$$

Base blocks: $\{0,1,3,5\}$, $\{0,1,4,7\}$, $\{0,1,5,7\}$, $\{0,1,5,8\}$, $\{0,1,9,11\}$ $\pmod{13}$.

Now let $w = 13$, $\lambda_1 = 1$ and $\lambda_2 = 4$ in Theorem 1, then for every $v \equiv 1$ or 4 $\pmod{12}$, $v \geq 55$, there exists a simple $(v, 13; \{4\}, 5)$-IPBD. Let $(X, Y, \mathcal{A})$ be such a simple $(v, 13; \{4\}, 5)$-IPBD. Form an indecomposable $NB(4,5;13)$ on Y, then we obtain an indecomposable $NB(4,5;v)$.

Let $v = 13$ in Lemma 5, we obtain an indecomposable $NB(4,5;40)$ from an indecomposable $NB(4,5;13)$. Let $v = 13$ in Lemma 4, we obtain an indecomposable $NB(4,5;49)$ and an indecomposable $NB(4,5;52)$. As $v \equiv 1$ or 4 $\pmod{12}$ and $v \geq 13$ is a necessary condition for the existence of an indecomposable $NB(4,5;v)$, the conclusion then follows. ∎

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