Concavity properties of numbers of solutions of Diophantine equations

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1. Introduction.

Many sequences encountered in the study of combinatorics and number theory are known to be log-concave. Although the notion of log-concavity is a fairly simple one, it is often very difficult to prove that a sequence has this property. It is easy to show that the sequence of binomial coefficients $\{\binom{n}{k} : n = k, k+1, \cdots\}$ is log-concave. In [5], Leib uses Newton's identities to prove that the Stirling numbers are log-concave. Carlitz [3] and Kurtz [4] use recurrence conditions to establish the property, and in [1] Ahuja and Enneking use methods from classical analysis. Brenti in [2] uses the theory of total positivity to study log-concave, unimodal, or Polya frequency sequences. Stanley in [8] gives a survey of various techniques.

In this paper we look at a problem that apparently has not been much studied. We consider sequences $\{a_n\}$ whose n-th term is the number of non-negative solutions to a given linear Diophantine equation with positive coefficients. Our main result is, that if at least three of the coefficients are equal to 1, then the corresponding sequence is eventually log-concave; the method of proof rests on finding appropriate polynomial upper and lower bounds for the sequence. Special cases not covered by the main result, are treated ad hoc.

Before proceeding further, we give some notation and terminology that will be used throughout the paper. A sequence $\{x_n : n = 0, 1, \cdots\}$ of positive real numbers is said to be *log-concave* (LC) provided $x_n^2 \geq x_{n-1}x_{n+1}$ for all $n = 1, 2, \cdots$. If $\{x_n : n = N, N+1, \cdots\}$ is LC for some $N \geq 0$, we say it is *eventually log-concave* (ELC). If an individual member of the sequence, say x_i, satisfies $x_i^2 \geq x_{i-1}x_{i+1}$, we say x_i is *locally log-concave* (LLC) in the sequence $\{x_n\}$, or just LLC if the sequence is understood. If $\alpha_1, \cdots, \alpha_m$ are positive integers, the symbol $N(\alpha_1, \cdots, \alpha_m; k; n)$ denotes the number of non-negative solutions of the Diophantine equation

$$\sum_{i=1}^{m} \alpha_i x_i + \sum_{i=1}^{k} y_i = n$$

Finally, we use the abbreviation $\{N(\alpha_1, \cdots, \alpha_m; k; n)\}$ to denote the sequence $\{N(\alpha_1, \cdots, \alpha_m; k; n) : n = 0, 1, \cdots\}$.

For a given set of coefficients $\{\alpha_1, \cdots, \alpha_m\}$ we are interested in determining whether $\{N(\alpha_1, \cdots, \alpha_m; k; n)\}$ is ELC. For certain cases, this question is easily answered. For example, it is clear that the sequence $\{N(2; 0; 1)\} = \{1, 0, 1, 0, \cdots\}$ is not ELC. On the other hand, the number of solutions to the equation $x_1 + \cdots + x_m = n$ is the binomial coefficient $\binom{n+m-1}{m-1}$ and it is known that the sequence $\{\binom{n+m-1}{m-1} : n = 0, 1, \cdots\}$ is LC for each $m \geq 1$. As we shall see, the answer has much to do with the number of the α_i that are equal to 1.

In § 2, we show that, for $k \geq 3$, the sequence $\{N(\alpha_1, \cdots, \alpha_m; k; n)\}$ is always ELC, and we give a sufficient condition for the ELC property to hold when $k = 2$. In § 3 we examine the special cases of linear diophantine equations with positive coefficients, in which at most two of the coefficients are different from 1, and we give a characterization of those that are ELC. Finally, in § 4, we show that for each set of positive coefficients $\{\alpha_1, \cdots, \alpha_m\}$, there exists an integer $K = K(\alpha_1, \cdots, \alpha_m)$ such that $\{N(\alpha_1, \cdots, \alpha_m; K; n)\}$ is LC. A counterexample shows that there is no single integer K which works for all choices of $\{\alpha_1, \cdots, \alpha_m\}$.

2.

In this section we show that the ELC property holds for a large class of sequences $\{N(\alpha_1, \cdots, \alpha_m; k; n)\}$. We will need the following two lemmas.

Lemma 2.1.

(a) *For $k \geq 1$ and $n \geq 0$,*

$$N(\alpha_1, \cdots, \alpha_m; k; n) = \sum_{i=0}^{n} N(\alpha_1, \cdots, \alpha_m; k-1; i)$$

(b) *For $k, n \geq 1$,*

$$N(\alpha_1, \cdots, \alpha_m; k; n) = N(\alpha_1, \cdots, \alpha_m; k-1; n) + N(\alpha_1, \cdots, \alpha_m; k; n-1)$$

Proof: The first equation follows by observing that the number of solutions to the equation $\alpha_1 x_1 + \cdots + \alpha_m x_m + y_1 + \cdots + y_k = n$ is equal to $\sum_{i=1}^{n} N(\alpha_1, \cdots, \alpha_m; k-1; n-i)$. The second equation follows immediately from (a). ∎

The next lemma may be found, in one form or another, in many elementary textbooks. We give it here in a convenient form.

Lemma 2.2. *Let a, b, and n be positive integers. Write $n = Q(ab) + R$ where $0 \leq R < ab$, and $R = qb + r$ where $0 \leq r < b$. Then $N(a, b; 0; R)$ is 0 or 1, and*

$$N(a, b; 0; n) = Q + N(a, b; 0; R).$$

Proof: $N(a,b;0;n)$ is the number of integers z such that $bz+r\equiv 0 \pmod a$ and $0\le z\le (n-r)/b = Qa+q$; since $N(a,b;0;R)$ is the number of these z in $[0,q]$, then $N(a,b;0;n) = N(a,b;0;R)+Q$. Since $0\le qb+r<ab$, then $q<a$, and it follows (since $(a,b)=1$) that $bz+r\equiv 0 \pmod a$ has precisely one solution t in $[0,a-1]$. Then $N(a,b;0;R)$ is 0 or 1, according as $t>q$, or $t\le q$. ∎

The next theorem shows that sequences $\{a(n,k) : n=0,1,2,\cdots\}$ satisfying the recurrence $a(n,k)=a(n,k-1)+a(n-1,k)$ (as in Lemma 2.1(b)), and that also have a certain type of polynomial growth, are ELC sequences. A proof appears in [5].

Theorem 2.3. *Let* $\{a(n,k) : n,k=0,1,2,\cdots\}$ *satisfy:*

(i) *for all* n, $a(n,0)$ *is a non-negative integer,*
(ii) *for all* k, $a(0,k)=c_0>0$,
(iii) *for all* $n,k>0$, $a(n,k)=a(n-1,k)+a(n,k-1)$.

Let k_0 *be fixed and suppose there exist polynomials* $P(x)$ *and* $Q(x)$ *of the same degree* d, *with respective leading coefficients* p *and* q *such that for all but finitely many* i, $P(i)\le a(i+1,k_0)\le Q(i)$. *Furthermore, suppose that* $(d+2)p^2\le (d+1)q^2$. *Then* $\{a(n,k_0+2) : n=0,1,2,\cdots\}$ *is ELC.* ∎

We now apply Theorem 2.3 to establish the ELC property for the numbers of solutions of linear Diophantine equations.

Theorem 2.4. *Let* $m>1$. *If there exist integers* $s\neq t$ *such that* $(\alpha_s,\alpha_t)=1$, *then* $\{N(\alpha_1,\cdots,\alpha_m;2;n)\}$ *is ELC.*

Proof: Using Lemma 2.1(b) and the fact that $\{N(\alpha_1,\cdots,\alpha_m;k;0)\}=1$ for all k, it suffices, by Theorem 2.3, to show that there exist polynomials $P(x)$ and $Q(x)$, both of degree $m-1$, having the same leading coefficients, such that

$$P(n)\le \{N(\alpha_1,\cdots,\alpha_m;0;n)\}\le Q(n) \tag{1}$$

for all n.

We prove (1) by induction on m. If $m=2$, then (1) holds by Lemma 2.2. Assume $m\ge 2$ and that (1) holds for m whenever there exist integers $s\neq t$ such that $(\alpha_s,\alpha_t)=1$. To prove the result is true for $m+1$, we may assume $(\alpha_1,\alpha_2)=1$. Note that for each $n\ge 0$,

$$N(\alpha_1,\cdots,\alpha_{m+1};0;n)=\sum_{i=0}^{[n/\alpha_{m+1}]} N(\alpha_1,\cdots,\alpha_m;0;n-i\alpha_{m+1})$$

where $[\cdots]$ denotes the greatest integer function. Denote this quantity by $\Gamma(n)$. By the induction hypothesis, there exist an integer a, and polynomials R and S

such that for all $i \in \{0, 1, \cdots [n/\alpha_{m+1}]\}$,

$$\begin{aligned} a(n-i\alpha_{m+1})^{m-1} + R(n-i\alpha_{m+1}) &\leq N(\alpha_1, \cdots, \alpha_m; 0; n-i\alpha_{m+1}) \\ &\leq a(n-i\alpha_{m+1})^{m-1} + S(n-i\alpha_{m+1}) \end{aligned} \tag{2}$$

where the degrees of R and S are less than $m-1$. Summing (2) over i we obtain

$$f(n) \leq \Gamma(n) \leq g(n)$$

where

$$f(x) = \sum_{i=0}^{[x/\alpha_{m+1}]} a(x - i\alpha_{m+1})^{m-1} + \sum_{i=0}^{[x/\alpha_{m+1}]} R(x - i\alpha_{m+1})$$

and

$$g(x) = \sum_{i=0}^{[x/\alpha_{m+1}]} a(x - i\alpha_{m+1})^{m-1} + \sum_{i=0}^{[x/\alpha_{m+1}]} S(x - i\alpha_{m+1})$$

(such sums may be thought of as upper or lower sums for corresponding integrals; thus the result of summation is a polynomial). Since R and S have degree at most $m-2$, it is clear that f and g are polynomials of degree m, with the same leading coefficient. ∎

The following corollary is immediate.

Corollary 2.5.

(i) *If $j \geq 3$, then $\{N(\alpha_1, \cdots, \alpha_m; j; n)\}$ is ELC.*

(ii) *If at least three members of $\{\alpha_1, \cdots, \alpha_m\}$ are equal to 1, then $\{N(\alpha_1, \cdots, \alpha_m; 0; n)\}$ is ELC.* ∎

Remark: As we shall see in the next section, if $j \leq 2$, there are many examples where $\{\alpha_1, \cdots, \alpha_m\}$ contains exactly j ones, but $\{N(\alpha_1, \cdots, \alpha_m; j; n)\}$ is not ELC. Thus, three is the least number of 1's guaranteeing the ELC property.

3.

In this section we study the ELC property of $\{N(\alpha_1, \cdots, \alpha_m; k; n)\}$ for the cases where at most two of the coefficients of the corresponding Diophantine equation are greater than 1. Thus we will be considering sequences having one of the forms $\{N(a, b; k; n)\}$ or $\{N(a; k; n)\}$, with the assumption that $a \neq 1$ and $b \neq 1$. Of course, by Corollary 2.5, $\{N(a, b; k; n)\}$ and $\{N(a; k; n)\}$ are ELC if $k \geq 3$, so we are interested in what happens if $k < 3$.

The sequences $\{N(a; k; n)\}$ with $k \leq 2$ are handled fairly easily. We first state a simple lemma.

Lemma 3.1. *Suppose that a, b, c are integers in arithmetic progression. Then* $b^2 - ac \geq 0$.

Proof: By assumption, there exists an integer r so that $b = a + r$, and $c = a + 2r$. Then $b^2 - ac = r^2 \geq 0$. ∎

Theorem 3.2. *Let* $a \geq 2$.

(i) $\{N(a;0;n)\}$ *and* $\{N(a;1;n)\}$ *are not ELC.*
(ii) $\{N(a;2;n)\}$ *is ELC if and only if* $a = 2$.
(iii) $\{N(a;2;n)\}$ *is LC.*

Proof: Note that $N(a;0;n) = 1$ if $a|n$, and is 0 otherwise. It follows from Lemma 2.1(a) that $N(a;1;n) = [n/a] + 1$, which is not LLC if $n \equiv -1 \pmod{a}$, and so (i) is true.

To prove (ii), we again apply Lemma 2.1 to get

$$N(a;2;n) = n + 1 + \sum_{i=0}^{n} [i/a].$$

Suppose first that $n \not\equiv -1 \pmod{a}$. Then

$$\begin{aligned} N(a;2;n+1) - N(a;2;n) &= N(a;1;n+1) \\ &= 1 + [(n+1)/a] = 1 + [n/a] \\ &= N(a;1;n) = N(a;2;n) - N(a;2;n-1). \end{aligned}$$

Hence the quantities $N(a;2;n-1), N(a;2;n), N(a;2;n+1)$ are in arithmetic progression, and by Lemma 3.1, $N(a;2;n)$ is LLC. Suppose, on the other hand, that $n = ja - 1$ for some $j \in \mathbf{Z}^+$. Then

$$N(a;2;n) = n + 1 + \sum_{i=0}^{n} [i/a] = aj + \sum_{k=0}^{j-1} ak = aj(j+1)/2.$$

and we have

$$\begin{aligned} &(3) \qquad & N(a;2;n-1) &= N(a;2;n) - N(a;1;n) \\ && &= N(a;2;n) - j, \\ &(4) \qquad & N(a;2;n+1) &= N(a;2;n) + j + 1. \end{aligned}$$

From (3) and (4), we have that $N(a;2;n)^2 - N(a;2;n-1)N(a;2;n+1)$ is equal to $j^2 + j - N(a;2;n) = j(j+1) - aj(j+1)/2$. Since a is an integer and $a \geq 2$, this quantity is non-negative if and only if $a = 2$. Thus $\{N(a;2;n)\}$ is LC if $a = 2$, but fails to be ELC if $a \geq 3$. ∎

We next direct our attention to $N(a,b;k;n)$. The following elementary lemma will be needed.

Lemma 3.3. *Let $a, b > 1$, and $(a, b) = 1$, and put $t = ab$. Denote $N(a, b; 0; n)$ by $c(n)$. Then for all $j > 0$, we have: $c(jab - 1) = j$, $c(jab) = j + 1$, $c(jab + 1) = j$, and $c(jab + 2) = j$ or $j + 1$.*

Proof: We have $c(0) = 1$, and since $a, b > 1$, $c(1) = 0$, and $c(2) = 0$ or 1, and $c(ab - 1) = 1$. Then the result follows immediately from Lemma 2.2. ∎

Theorem 3.4. *Let $a, b > 1$. Then $\{N(a, b; 0; n)\}$ and $\{N(a, b; 1; n)\}$ are not ELC.*

Proof: Using the values given by Lemma 3.3, $N(a, b; 0; jab + 1)$ is not LLC. Thus $\{N(a, b; 0; n)\}$ is not ELC.

Next, let $c(n) = \{N(a, b; 1; n)\}$ and put $(a, b) = d$, $t = ab/d^2$, $\alpha = a/d$, and $\beta = b/d$. For any positive i, we have $N(a, b; 0; id) = N(\alpha, \beta; 0; i)$. By Lemma 2.1, $c(n) - c(n-1) = N(a, b; 0; n)$, and it follows that $c(n) = N(\alpha, \beta; 1, [n/d])$ (so that the sequence $\{N(a, b; 1; n)\}$ consists of consecutive constant "blocks" of length d). If $d > 1$, then for any $i > 1$, $c(id - 2) = c(id - 1) = c(id) - 1$, and it follows that $c(id - 1)$ is not LLC, and hence the sequence $\{c(n)\}$ is not ELC.

Suppose now that $d = 1$. We will show that for all $i \geq 2$,

$$(c(it - 1))^2 < c(it)c(it - 2).$$

As before, from Lemmas 2.1 and 3.3, we have

$$c(it) - c(it - 1) = i + 1, \quad c(it - 1) - c(it - 2) = i.$$

It suffices to prove that $(c(it - 1))^2 < (c(it - 1) + i + 1)(c(it - 1) - i)$, which holds if and only if $c(it - 1) > i(i + 1)$. From Lemma 2.2 we have $N(a, b; 0; n) \geq n/t$, and then by Lemma 2.1,

$$c(it - 1) \geq \sum_{j=0}^{it-1} [j/t] > \sum_{j=0}^{it-1} j/t = i(it - 1)/2,$$

and then, noting that $d = 1$ and $a, b > 1$ imply that $t \geq 6$, it follows that $c(it - 1) > i(i + 1)$. This completes the proof. ∎

So we know that $\{N(a, b; 1; n)\}$ is never ELC if $a, b > 1$, while $\{N(a, b; 3; n)\}$ is always ELC. The question of when $\{N(a, b; 2; n)\}$ is ELC is answered in the next theorem. We will need the following lemma.

Lemma 3.5. *Let $t \in \mathbf{Z}^+$ and $d \geq 2$. Let $\{b_0, b_1, \cdots\}$ be a sequence of positive integers satisfying $b_0 = 1$, and $b_n \leq b_{n-1} + j$ whenever $(j - 1)t \leq n \leq jt - 1$. Then for all $j \in \mathbf{Z}^+$ and all n such that $(j - 1)t \leq n \leq jt - 1$,*

$$b_n(b_n + j + 1) < (j + 1)d \sum_{i=0}^{n} b_i \tag{5}$$

Proof: The proof is by induction on n. If $n = 0$, then $j = 1$, so that (5) holds. Now assume the inequality holds for n. We will consider two cases. Suppose first that n is such that $(j-1)t \leq n+1 \leq jt-1$ for some $j \in \mathbf{Z}^+$. We need to show that

$$(j+1)d\sum_{i=0}^{n+1} bi > b_{n+1}(b_{n+1}+j+1).$$

By the induction hypothesis, and using the fact that $b_{n+1} < b_n + j$, we get

$$\begin{aligned}(j+1)d\sum_{i=0}^{n+1} bi &> b_n(b_n+j+1)+(j+1)db_{n+1} \\ &> (b_{n+1}-j-1)b_{n+1}+(j+1)db_{n+1} \\ &> b_{n+1}(b_{n+1}+(d-1)(j+1)) \\ &> b_{n+1}(b_{n+1}+j+1)\end{aligned}$$

since $d \geq 2$.

Now suppose that $n = jt-1$. We need to show that

$$(j+2)d\sum_{i=0}^{n+1} b_i > b_{n+1}(b_{n+1}+j+2).$$

As before,

$$\begin{aligned}(j+2)d\sum_{i=0}^{n+1} bi &> (b_n+j+1)b_n + d\sum_{i=0}^{n} b_i + (j+2)db_{n+1} \\ &> b_{n+1}(b_{n+1}-j-1) + d\sum_{i=0}^{n} b_i + (j+2)db_{n+1} \\ &= b_{n+1}(b_{n+1}-j-1+d(j+2)) + \sum_{i=0}^{n} b_i \\ &> b_{n+1}(b_{n+1}+j+2)\end{aligned}$$

which completes the proof. ∎

Theorem 3.6. *Let $a, b > 1$, with $(a,b) = d$. Then $\{N(a,b;2;n)\}$ is ELC if and only if $d = 1$. In particular, if $d > 1$, then for all $j \in \mathbf{Z}^+$, $N(a,b;2;djt-1)$ is not LLC in $\{N(a,b;2;n)\}$, where $t = ab/d^2$.*

Proof: If $d = 1$, then $\{N(a,b;2;n)\}$ is ELC by Theorem 2.4. So assume $d > 1$, fix j, and put $r = djt$. Let $c(i) = N(a,b;1;i)$, and put $T = N(a,b;2;r-1)$.

By Lemma 2.1(b), it suffices to show that $T^2 - (T - c(r-1))(T + c(r)) < 0$, i.e., that

$$(6) \qquad T(c(r) - c(r-1)) > c(r)c(r-1).$$

We use Lemma 3.5. Let $\alpha = a/d$, $\beta = b/d$, and $t = ab/d^2 = \alpha\beta$. As noted in Theorem 3.4, $c(n) = N(\alpha, \beta, 1, [n/d])$, and by Lemma 3.3, $c(r) - c(r-1) = j + 1$. By Lemma 2.1, we have

$$T = d \sum_{i=0}^{jt-1} N(\alpha, \beta; 1; i).$$

From Lemma 2.1, $N(\alpha,\beta;1;n) - N(\alpha,\beta;1;n-1) = N(\alpha,\beta;0;n)$, and from Lemma 2.2, $N(\alpha,\beta;0;k) = [k/\alpha\beta] + (0 \text{ or } 1)$. Thus, since $t = \alpha\beta$, then for all integers n such that $jt \leq n \leq (j+1)t - 1$ for some j, we have

$$N(\alpha,\beta;1;n) - N(\alpha,\beta;1;n-1) = N(\alpha,\beta;0;n) = j \text{ or } j+1.$$

Now the conditions of Lemma 3.5 are satisfied, and the inequality (6) follows. ∎

4.

In this section, we shall prove that for each set of positive coefficients $\{\alpha_1, \cdots, \alpha_m\}$, there exists an integer $K = K(\alpha_1, \cdots, \alpha_m)$ such that $\{N(\alpha_1, \cdots, \alpha_m; K; n)\}$ is LC. It will be convenient to use generating functions. It is clear that the generating function for $\{N(\alpha_1, \cdots, \alpha_m; 0; n)\}$ is given by

$$f(\alpha_1, \cdots, \alpha_m) = \prod_{i=1}^{m} (1 - x^{\alpha(i)})^{-1}$$

where we write $\alpha_i = \alpha(i)$, which is the product (or convolution) of the generating functions for $\{N(\alpha_1; 0; n)\}, \{N(\alpha_2; 0; n)\}, \cdots, \{N(\alpha_m; 0; n)\}$. The generating function for $\{N(\alpha_1, \cdots, \alpha_m; k; n)\}$ is $(1-x)^{-k} f(\alpha_1, \cdots, \alpha_m)$. It is well known that if two or more sequences are LC, then their convolution is also LC (see, e.g. [2]), and we use this in what follows. First, we give a counterexample to show that there is no fixed value K such that the sequences $\{N(\alpha_1, \cdots, \alpha_m; K; n)\}$ are LC for all choices of positive integers $\alpha_1, \cdots, \alpha_m$.

Example 4.1: Let m be a positive integer, and put $\alpha_i = 2$ $(i = 1, 2, \cdots, m)$. For all $n, k \geq 0$, put $a(n,k) = N(\alpha_1, \cdots, \alpha_m; k; n)$. Then we have

$$a(0,k) = 1; \quad a(1,k) = k; \quad a(2,k) = m + \binom{k+1}{2},$$

and $a(1,k)^2 - a(0,k)a(2,k) = (k^2 - 2m - k)/2$; this is negative if m is greater than $k(k-1)/2$. Then for every value of k, we can find an m so that $\{N(\alpha_1, \cdots, \alpha_m; k; n)\}$ (with $\alpha_i = 2$ $(i = 1, 2, \cdots, m)$) is not LC.

The main result of this section is, that for a fixed choice of $\alpha_1, \cdots, \alpha_m$, the sequence $\{N(\alpha_1, \cdots, \alpha_m; 3m; n)\}$ is LC. This is proved first for the case $m = 1$. We begin with some preliminary lemmas.

The following notation will be used throughout the case for $m = 1$.

$$\alpha > 1 \text{ is an integer,}$$
$$a_n = N(\alpha; 0; n) \qquad b_n = N(\alpha; 1; n)$$
$$c_n = N(\alpha; 2; n) \qquad d_n = N(\alpha; 3; n)$$

Lemma 4.2.

(i) $a_n = 1$ *if* $\alpha | n$, *and is* 0 *otherwise.*
(ii) $b_n = 1 + [n/\alpha]$.

Proof: (i) is obvious, and (ii) follows immediately from Lemma 2.2. ∎

The next lemma gives some explicit summations involving the greatest integer function. The proofs of these formulas are straightforward, and we omit them.

Lemma 4.3. *For a non-negative integer n, put $r = n - \alpha[n/\alpha]$, and $t = [n/\alpha]$. Then*

(i) $$\sum_{j=0}^{n} [j/\alpha] = (\alpha/2)t(t+1) + (1+r)t + r - n.$$

(ii) $$\sum_{j=0}^{n} j[j/\alpha] = t\binom{n+1}{2} + (\alpha/2)\binom{t+1}{2} - (\alpha^2/8)\binom{2t+2}{3}$$

(iii) $$\sum_{j=0}^{n} [j/\alpha]^2 = (\alpha/4)\binom{2t}{3} + t^2(r+1).$$

∎

Lemma 4.4. *With r and t defined as in Lemma 4.3,*

$$c_n = (t+1)(n+r+2)/2,$$
$$d_n = (\alpha/6)\binom{t}{2}(2\alpha t - \alpha + 3) + t(1+r)(n - \alpha + 2)/2 + \binom{n+2}{2}$$

Proof: We have $c_n = \sum_{i=0}^{n} b_n$ and $d_n \sum_{i=0}^{n} c_n$. Lemma 4.3(i) then allows the evaluation of c_n, and then using Lemma 4.3 (ii) and (iii), we compute d_n. ∎

Theorem 4.5. *For all integers* $n > 0$ *and* $\alpha > 1$, *and with* $r = r(n, \alpha)$ *and* $t = t(n, \alpha)$ *defined as above,* $d_{n+1}^2 - d_n d_{n+2} > 0$.

Proof: If $\alpha = 2$, then by Theorem 3.2(iii), $\{c_n\}$ is LC, and then $\{d_n\}$ is also LC since it is a convolution of LC sequences. Thus we suppose that $\alpha > 2$. Let n and $\alpha > 2$ be fixed; so that $t = [n/\alpha]$ and $r = n - \alpha t$. By a routine computation, we have

$$d_{n+1}^2 - d_n d_{n+2} = c_{n+1}^2 - b_{n+2} d_n.$$

Define $\delta(i, \alpha)$ to be 1 if α divides i, and 0 otherwise. Then

$$b_{n+2} = t + 1 + \delta(n+1, \alpha) + \delta(n+2, \alpha)$$
$$c_{n+1} = (t+1) + (t+1)(n+r+2)/2 + \delta(n+1, \alpha).$$

We consider four cases.

Case 1: Suppose that $\alpha | n$. Then $r = 0$, $n = \alpha t$, $b_{n+2} = t + 1$, $c_{n+1} = (t+1) + (t+1)(n+2)/2$, and by a routine computation, we have

$$d_{n+1}^2 - d_n d_{n+2} =$$
$$\{t^2(n^2 + 15n + 36) + t(n^2 + 30n + 72) + (-n^2 + 15n + 36) - n\alpha\}/12$$

This expression is clearly positive if $t > 0$, and since $n = \alpha t$, if $t = 0$, then $n = 0$ also, and then the expression reduces to 3.

Case 2: Suppose that $\alpha | (n+1)$. Then $n = \alpha t + (\alpha - 1)$, $r = \alpha - 1$, $b_{n+2} = t + 2$, and $c_{n+1} = (t+2) + (t+1)(n + \alpha + 1)$; we compute

$$d_{n+1}^2 - d_n d_{n+2} =$$
$$\{t^2(n^2 + 11n + 22) + t(3n^2 + 42n + 87) + (n^2 + 38n + 85) - (n+1)\alpha\}/12$$

In this case, $n + 1 = \alpha(t+1)$, so if $t = 0$, the expression reduces to $3\alpha + 4$, which is positive. If $t > 0$, we have $n^2 - (n+1)\alpha = (t+1)\alpha(\alpha t - 2) + 1$, which is positive, and so the whole expression is positive.

Case 3: Suppose $\alpha | (n+2)$. Then in a similar way, we get

$$d_{n+1}^2 - d_n d_{n+2} =$$
$$\{t^2(n^2 + 7n + 10) + t(3n^2 + 24n + 36) + (n^2 + 16n + 78) - (n+2)\alpha\}/12$$

which is easily seen to be positive.

Case 4: Suppose that α does not divide any of $n, n+1, n+2$. Then $n = \alpha t + r$, where $0 < r \leq \alpha - 3$, and we find that

$$d_{n+1}^2 - d_n d_{n+2} =$$
$$(t+1)\{t(n^2+15n+36)+(15n+36)+r(4nt+15t+6n+15)+r^2 t-(n-r)\alpha\}/12$$

which is clearly positive. ∎

We can now state the main result of this section.

Theorem 4.6. *The sequence* $\{N(\alpha_1, \cdots, \alpha_m; 3m; n)\}$ *is LC.*

Proof: For each i, $1 \leq i \leq m$, the sequence $\{N(\alpha_i; 3; n)\}$ is LC, and the corresponding generating function is $(1-x)^{-3} f(\alpha_i)$. Then the sequence $\{N(\alpha_1, \cdots, \alpha_m; 3m; n)\}$ has generating function $(1-x)^{-3m} f(\alpha_1, \cdots, \alpha_m)$, and must be LC since it is the convolution of LC sequences. ∎

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