Maximal Configurations of Stars

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Abstract. A star S_q with q edges is a complete bipartite graph $K_{1,q}$. Two figures of the complete graph K_n on a given set of n vertices are compatible if they are edge-disjoint, and a configuration is a set of pairwise compatible figures. In this paper, we take stars as our figures. A configuration C is said to be maximal if there is no figure (star) $f \notin C$ such that $\{f\} \cup C$ is also a configuration. The size of a configuration F, denoted by $|F|$, is the number of its figures.

Let Spec(n,q) (or simply Spec(n)) denote the set of all sizes such that there exists a maximal configuration of stars with this size. In this paper, we completely determine Spec(n), the spectrum of maximal configurations of stars. As a special case, when n is an order of a star system, we obtain the spectrum of maximal partial star systems.

1. Introduction.

Let S_q be a star on $q + 1$ vertices. A complete graph K_n is said to have a G-decomposition $G[n]$ if it is a union of edge-disjoint subgraphs of K_n each of which is isomorphic to a fixed graph G. The basic problem connected with the G-decomposition is to determine, for a given graph G, the necessary and sufficient conditions on n for the existence of a decomposition $G[n]$. When the graph G is itself a complete graph K_k, then the decomposition $K_k[n]$ is known as a balanced incomplete block design (BIBD) [1]. For the case where G is a star, the problem is completely settled by M. Tarsi in a more general way [3]. As a special case, he proved that the necessary and sufficient conditions for the existence of an S_q-decomposition are that $n \geq 2q$ and $n(n-1) \equiv 0 \pmod{2q}$. This decomposition may also be referred as an (n, q) star system (or simply a star system). If we start with a fixed q, then for the nunmber n which does not satisfy the necessary and sufficient conditions mentioned above we can consider the problem of packing K_n with as many stars, S_q as possible.

Two figures of the complete graph K_n on a given set of n vertices are compatible if they are edge-disjoint, and a configuration is a set of pairwise compatible figures. In what follows, our configuration will be a set of pairwise edge-disjoint stars of the complete graph K_n. A configuration C is said to be maximal if there is no star $f \notin C$ such that $\{f\} cupC$ is also a configuration. The size of a configuration F, denoted by $|F|$, is the number of its stars. Obviously, if $|F| = \lfloor \binom{n}{2}/q \rfloor$, then F

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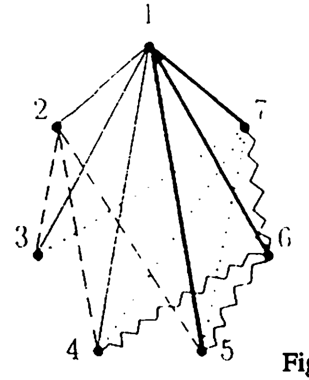
is maximal. We are also interested in configurations which are maximal but have size less than $\lfloor \binom{n}{2}/q \rfloor$. Thus, let Spec($n, q$) or simply Spec($n$) denote the set of all sizes such that there exists a maximal configuration of stars with this size. In this paper, we completely determine Spec(n), the spectrum of maximal configurations of stars. As a special case, when n is an order of a star system, we obtain the spectrum of maximal partial star systems.

For the rest of this paper, without mentioning otherwise, we consider configuration of K_n, where $n = mq + s$, $1 \leq s \leq q$, in which the figures are stars. Since there is no star S_q in K_s, we also assume $m \geq 1$. In what follows, we will use a $(q+1) \times b$ array $A = [a_{ij}]$ to represent a configuration with b stars where the first row represents the vertices of the centers and the degree one vertices of the jth star are $a_{2j}, a_{3j}, \dots, a_{(q+1)j}$. Figure 2.1 is an example of such a representation.



$$A = \begin{bmatrix} 1 & 1 & 2 & 6 & 7 \\ 2 & 5 & 3 & 4 & 3 \\ 3 & 6 & 4 & 5 & 4 \\ 4 & 7 & 5 & 7 & 5 \end{bmatrix}$$

Figure 2.1

Let $A = [a_{ij}]$ be an array which represents a configuration F. Then, by observation, the degree of a vertex v in F, $\deg_F(v)$, can be obtained as $\alpha q + \beta$ where α is the number of a_{1j}, $j = 1, 2, \dots, b$, such that $a_{1j} = v$ and β is the number of a_{ij} which is equal to v, $i = 2, 3, \dots, q+1$ and $j = 1, 2, \dots, b$. Now, consider a configuration F on n vertices. If for each vertex v, the degree of v in the graph $K_n \backslash F$, the complement of F, is less than q, then F is a maximal configuration, that is, if $\deg_F(v)$ is greater than $n - 1 - q$, we conclude that F is a maximal configuration. Since, we will use this result often, we list it as a lemma.

Lemma 1.1. *In a configuration F, if $\deg_F(v) \geq n - q$ for each v in $V(K_n)$, then F is maximal and the number of columns of an array which represents F is in Spec(n). Conversely, if F is a configuration in which there is a vertex of degree 1 less than $n - q$, then F is not a maximal configuration.*

Corollary 1.2. min *Spec(n)* $\geq \left\lceil \frac{n(n-q)}{2q} \right\rceil$.

What we try to prove in this paper is the following

Theorem 1.3. *(i)* $Spec(n) = \{s, s+1, \ldots, 2s-1\}$ *if* $n = q+s$ *and* $1 \le s \le q$.

(ii) $Spec(n) = \left\{x\colon \left\lceil \frac{n(n-q)}{2q} \right\rceil \le x \le \left\lfloor \frac{n(n-1)}{2q} \right\rfloor \right\}$ *if* $n \ge 2q+1$.

2. The main results.

We will start our construction with the smaller n. (We note here that the maximal configuration of size min Spec(n) is just the smallest possible value of our construction.)

First, consider $n = q + s$. Let $b = s + t$, $\delta = q - t$, and $\gamma = \lfloor \frac{b-1}{2} \rfloor$ where $1 \le s \le q$ and $0 \le t \le q$. Also, let $V(K_n) = \{1, 2, 3, \ldots, b, v_1, v_2, \ldots, v_\delta\}$ and construct two arrays B and C as in Figure 2.2.

$$B\colon \begin{bmatrix} 2 & 3 & 4 & \cdots & b & 1 \\ 3 & 4 & 5 & \cdots & 1 & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \gamma+1 & \gamma+2 & \gamma+3 & \cdots & \gamma-1 & \gamma \end{bmatrix} \quad C\colon \begin{bmatrix} v_1 & v_1 & v_1 & \cdots & v_1 \\ v_2 & v_2 & v_2 & \cdots & v_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ v_\delta & v_\delta & v_\delta & \cdots & v_\delta \end{bmatrix}$$

Figure 2.2

Now we are ready to construct an array which represents a maximal configuration. If $\lfloor \frac{s\delta}{b} \rfloor \ge q - \gamma$, then A_3 (Figure 2.3) can be defined as follows: $A_3(1, j) = j, 1 \le j \le b$; $A_3(i, j) = v_{x+1}$ if $b(i-2) + j - 1 = sx + r$ for some $0 \le r < s$, where $2 \le i \le q+1, 1 \le j \le b$, and $bi + j \le s\delta$; $A_3(i, j) = B(i + \gamma - q - 1, j)$ if $bi + j > s\delta$.

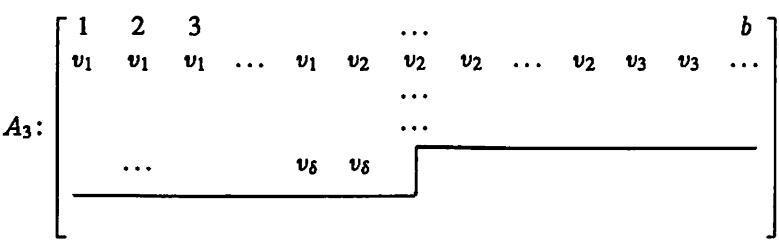


Figure 2.3

For the situation $\lfloor \frac{s\delta}{b} \rfloor < q - \gamma$, A_4 (Figure 2.4) can be defined similarly except that the whole of B is in the bottom of A_4; for each $x \le e$, v_x occurs $b' + 1$ times and v_y occurs b' times whenever $e < y \le \delta$. (b' is a number between s and b such that $(q - \gamma)b = \delta b' + e, 0 \le e < \delta$.)

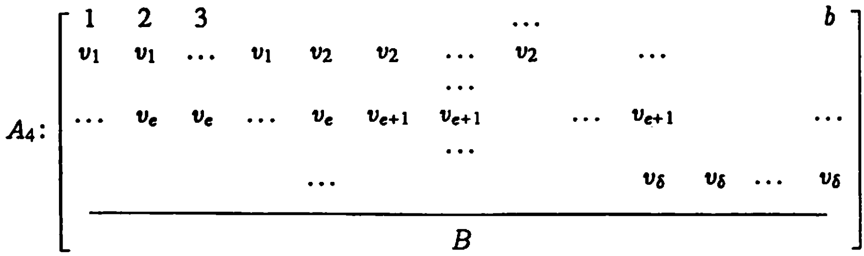


Figure 2 .4

It is not difficult to see that if $\delta \geq q-\gamma$, such an integer solution for b' exists. Hence $q-t=\delta \geq q-\gamma$, that is, $t \leq \lfloor \frac{s+t-1}{2} \rfloor$. This implies that $t \leq s-1$. Thus we can construct A_4 for each $0 \leq t \leq s-1$, and in the corresponding configuration F, $\deg_F(v) \geq s$, $v \in V(K_{q+s})$, so by Lemma 2.1 we have a maximal configuration of stars whenever $0 \leq t \leq s-1$. Equivalently, $\{s, s+1, \ldots, 2s-1\} \subseteq$ Spec$(q+s)$. Now, suppose $x \in$ Spec$(q+s)$, then the following two inequalities hold:

$$(1)\ xq \geq (q+s-x)s, \text{ and } (2)\ xq \leq \frac{x(x-1)}{2} + (q+s-x)x.$$

It is easy to see $s \leq x \leq 2s-1$. Hence we have the following

Proposition 2.2. *For* $1 \leq s \leq q$, *Spec*$(q+s) = \{s, s+1, \ldots, 2s-1\}$.

Next, if $n = 2q+s$, $1 \leq s \leq q$, and $\left\lceil \frac{s(q+s)}{2q} \right\rceil \leq t \leq q$. We note here that $q+s+\left\lceil \frac{s(q+s)}{2q} \right\rceil = \left\lceil \frac{(2q+s)(q+s)}{2q} \right\rceil$ = min Spec$(2q+s)$. Let $b=q+s+t$, $\delta = q-t$ and $\gamma = \lfloor \frac{b-1}{2} \rfloor$. B and C are defined as in Figure 2.2. We will construct A_5 in a similar way. First, if $\lfloor \frac{(q+s)\delta}{b} \rfloor \geq q-\gamma$, A_5 (Figure 2.5) can be defined as follows: $A_5(1,j)=j$, $1 \leq j \leq b$; $A_5(c,j) = v_{x+1}$ if $b(i-2)+j-1 = (q+s)x+r$ for some $0 \leq r < q+s$ where $2 \leq i \leq q+1$, $1 \leq j \leq b$, and $bi+j \leq (q+s)\delta$; $A_5(i,j) = B(i+\gamma-q-1, j)$ if $bi+j > (q+s)\delta$.

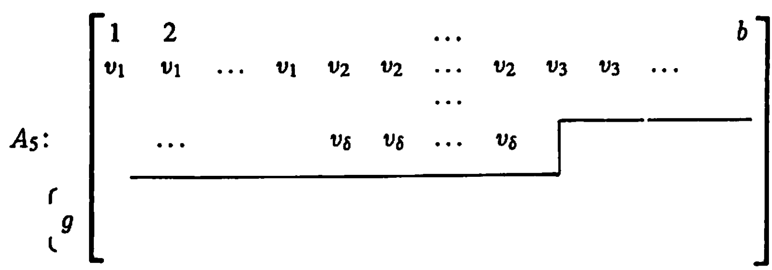


Figure 2.5

Let g be the number of rows in A_5 (Figure 2.5) which are below the rows containing the v_i's, $i = 1, 2, \ldots, \delta$. If $g < s$, then

$$(q+s)\delta+s(q+s+t) = (q+s)\delta+(s-g)(q+s+t)+g(q+s+t) > q(q+s+t).$$

This implies that $t < \frac{s(q+s)}{2q} \leq \left\lceil \frac{s(q+s)}{2q} \right\rceil$ which is a contradiction. Hence $g \geq s$, and we obtain a maximal configuration by Lemma 2.1. Secondly, if $\lfloor \frac{(q+s)\delta}{b} \rfloor < q - \gamma$, similar to the case $n = q + s$, the matrix is similar to A_4 except that $q + s < b' < b$. The fact that it represents a maximal configuration follows by a direct computation; we omit the details. So far, we have $\{q + s + \left\lceil \frac{s(q+s)}{2q} \right\rceil,$ $q + s + \left\lceil \frac{s(q+s)}{2q} \right\rceil + 1, \ldots, 2q + s\} \subseteq \text{Spec}\,(2q + s)$. For the large values, we need other constructions. Let $b = 2q + s - y$, $1 \leq y \leq s - 1$. By the array in Figure 2.6, it is not difficult to see that $b + 2y \in \text{Spec}\,(2q + s)$, that is, $\{2q + s + 1,\ 2q + s + 2, \ldots, 2q + 2s - 1\} \subseteq \text{Spec}(2q + s)$.

$$A_6: \left[\begin{array}{cccccc|ccccccc}
1 & 2 & 3 & \cdots & b-1 & b & v_1 & v_1 & v_2 & v_2 & \cdots & v_y & v_y \\
2 & 3 & 4 & \cdots & b & 1 & 1 & q+1 & 2q+1 & & & & \\
3 & 4 & 5 & \cdots & 1 & 2 & 2 & q+2 & 2q+2 & & \cdot & & \\
 & & & & & & & & \vdots & & & & \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & b & & & \cdot & \\
 & & & & & & & & 1 & & & & \\
 & & & & & & & & 2 & & & & \\
q+1 & q+2 & q+3 & \cdots & q-1 & q & q & 2q & \vdots & & & & \cdot
\end{array}\right]$$

Figure 2 .6

We still have some values left. Let us start with another array which represents a maximal configuration with size $2q + 2s - 1$. See Figure 2.7. (We omit the detail definition of A_7.)

$$A_7: \left[\begin{array}{cccccc|cccc|cccc}
1 & 2 & 3 & \cdots & 2q-2 & 2q-1 & v_1 & v_2 & \cdots & v_s & v_1 & v_2 & \cdots & v_s \\
2 & 3 & 4 & \cdots & 2q-1 & 1 & 1 & 1 & & 1 & q+1 & q+1 & \cdots & q+1 \\
3 & 4 & 5 & \cdots & 1 & 2 & 2 & 2 & \cdots & 2 & q+2 & q+2 & \cdots & q+2 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
q & q+1 & q+2 & \cdots & q-2 & q-1 & q-1 & q-1 & \cdots & q-1 & 2q-1 & 2q-1 & \cdots & 2q-1 \\
2q & 2q & 2q & \cdots & 2q & 2q & q & q & \cdots & q & 2q & 2q & \cdots & 2q
\end{array}\right]$$

Figure 2 .7

As can be seen in the array, we have already used up the degree of the vertices $1,2,\ldots,2q$, and we are going to adjust the degree of $v_1, v_2, \ldots, v_s$ in order to obtain a maximal configuration with larger size. From A_7, we have that in the representing configuration F, $\deg_F(v_i) = 2q$, $i = 1,2,\ldots,s$. Thus, $K_n \backslash F$ has $\binom{s}{2}$ edges. If $\binom{s}{2} < q$, then $2q + 2s - 1 = \max \text{Spec}(2q + s)$, we are done. If $\binom{s}{2} \geq q$, we can adjust A_7 by replacing the $2q-1$ edges $\{q+1,1\}, \{q+2,1\}, \ldots, \{2q-1,1\}, \{v_1,1\}, \{v_1,q+1\}, \ldots, \{v_1, q+s-2\}, \{v_2, q+s-2+1\}, \ldots, \{v_2, q+2(s-2)\}, \{v_3, q+2(s-2)+1\}, \ldots, \{v_3, q+2(s-2)+(s-3)\}, \ldots,$ and $\{v_i, q+\sum_{j=1}^{i}(s-j)+h-1\}$ (in order) with $\{q+1, v_1\}, \{q+2, v_1\}, \ldots, \{q+(s-2), v_1\}, \{q+s-2+1, v_2\}, \ldots, \{2q-1, v_i\}, \{v_1, v_2\}, \{v_1, v_3\}, \ldots, \{v_1, v_s\}, \{v_2, v_3\}, \ldots, \{v_2, v_s\}, \{v_3, v_4\}, \ldots$, and $\{v_i, v_{i+h}\}$ respectively, where the first element of the edge represents the center of the star in which this edge belongs, and $\sum_{j=1}^{i}(s-j) + h - 1 = q - 1$. In this way, we can add one more star with center 1 whose edges are $\{1, q+1\}, \{1, q+2\}, \ldots, \{1, 2q-1\}$ and $\{1, v_1\}$. This implies that $2q + 2s \in \text{Spec}(2q + s)$. Now if $\binom{s}{2} - q \geq q$, we can use a similar process to obtain a new star with center 2 except the new edges which we will use are those q edges $\{v_i, v_{i+h+1}\}, \ldots, \{v_i, v_s\}, \{v_{i+1}, v_{i+2}\}, \ldots, \{v_{i+1}, v_s\}, \ldots, \{v_j, v_{j+h'}\}$. (Replace $\{1,2\}, \{q+2,2\}, \{q+3,2\}, \ldots, \{2q-1,2\}, \{v_{i+h+1},2\}, \{v_{i+h+2},1\}, \{v_i, q+2\}, \ldots$, and $\{v_j, 2q-1\}$ (in order) with $\{1, v_{i+h+2}\}, \{q+2, v_i\}, \ldots, \{2q-1, v_j\}, \{v_i, v_{i+h+1}\}, \{v_i, v_{i+h+2}\}, \{v_i\ v_{i+h+3}\}, \ldots, \{v_i, v_s\}, \ldots,$ and $\{v_j, v_{j+h'}\}$, respectively.) We will stop this process whenever the number of edges which are not used is less than q. Since the same v_i will not occur in the same column which is not difficult to see, thus we have

Proposition 2.3. *For* $1 \leq s \leq q$,

$$Spec(2q+s) = \left\{ q+s+\left\lceil \frac{s(q+s)}{2q} \right\rceil, \right.$$
$$\left. q+s+\left\lceil \frac{s(q+s)}{2q} \right\rceil + 1, \ldots, 2q+2s-1+\left\lfloor \binom{s}{2} \Big/ q \right\rfloor \right\}.$$

On the case $n = 3q + s$, *let* $n' = 2q + s$. *Construct a maximal configuration* F *on* n' *vertices. Add* q *new vertices and* $2q + s$ *stars by joining each of the* n' *vertices to every new vertex. Then the new configuration is also maximal. Thus we have* $\min\ Spec(3q + s) = \min\ Spec(2q + s) + 2q + s$, *and* $2q + s + x \in Spec(3q + s)$ *for each* $x \in Spec(2q + s)$. *By Proposition 2.3 we conclude that*

$$\left\{ 3q+2s+\left\lceil \frac{s(q+s)}{2q} \right\rceil, 3q+2s+\left\lceil \frac{s(q+s)}{2q} \right\rceil + 1, \right.$$
$$\left. \ldots, 4q+3s-1+\left\lfloor \binom{s}{2} \Big/ q \right\rfloor \right\} \subseteq Spec(3q+s).$$

If we start with a maximal configuration of size $4q+3s-1$, *then we can follow a similar process as in the case* $n=2q+s$ *to obtain* $Spec(3q+s)$. *We will give the configuration of size* $4q+3s-1$ *and omit the details. (Figure 2.8.)*

Proposition 2.4. *For* $1 \le s \le q$,

$$Spec(3q+s) = \left\{3q+2s+\left\lceil\frac{s(q+s)}{2q}\right\rceil, 3q+2s+\left\lceil\frac{s(q+s)}{2q}\right\rceil+1, \ldots, \left\lfloor\frac{(3q+s)(3q+s-1)}{2q}\right\rfloor\right\}.$$

$$M_1: \left[\begin{array}{cccc|cccc|cccc} 1 & 2 & \ldots & 2q-1 & 2q+1 & 2q+2 & \ldots & 3q & 2q+1 & 2q+2 & \ldots & 3q \\ 2 & 3 & \ldots & 1 & 2q & 2q & \ldots & 2q & q & q & \ldots & q \\ 3 & 4 & \ldots & 2 & 1 & 1 & \ldots & 1 & q+1 & q+1 & \ldots & q+1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ q & q+1 & \ldots & q-1 & q-2 & q-2 & \ldots & q-2 & 2q-2 & 2q-2 & \ldots & 2q-2 \\ 2q & 2q & \ldots & 2q & q-1 & q-1 & \ldots & q-1 & 2q-1 & 2q-1 & \ldots & 2q-1 \end{array}\right]$$

$$M_2: \left[\begin{array}{cccc|cccc|cccc} v_1 & v_2 & \ldots & v_s & v_1 & v_2 & \ldots & v_s & v_1 & v_2 & \ldots & v_s \\ 2q & 2q & \ldots & 2q & q & q & \ldots & q & 2q+1 & 2q+1 & \ldots & 2q+1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ q-2 & q-2 & \ldots & q-2 & 2q-2 & 2q-2 & \ldots & 2q-2 & 3q-1 & 3q-1 & \ldots & 3q-1 \\ q-1 & q-1 & \ldots & q-1 & 2q-1 & 2q-1 & \ldots & 2q-1 & 3q & 3q & \ldots & 3q \end{array}\right]$$

$A_8: [\, M_1 \mid M_2 \,]$

Figure 2.8

Now, we are ready to consider $n = mq+s$, $m \ge 4$. *By a direct counting, we have*

$$\begin{aligned} \min\ Spec(n) &= \min\ Spec(3q+s) + (3q+s)(m-3) + \binom{m-3}{2}q \\ &= \min\ Spec(2q+s) + (2q+s)(m-2) + \binom{m-2}{2}q. \end{aligned}$$

As a special case of $n=q+s$, $Spec(2q) = \{q, q+1, \ldots, 2q-1\}$. *If* m *is even, then*

$$Spec(n) \supseteq \left\{x : x = y + (2q+s)(m-2) + \binom{m-2}{2}q - \frac{m-2}{2}\cdot q + z_1 + z_2 + \ldots + z_{\frac{m-2}{2}}, \text{ where } y \in Spec(2q+s) \text{ and } z_i \in Spec(2q), i = 1, 2, \ldots, \frac{m-2}{2}\right\}.$$

This implies that

$$Spec \supseteq \left\{ q+s+\left\lceil \frac{s(q+s)}{2q} \right\rceil + (2q+s)(m-2) + \binom{m-2}{2}q, \ldots , \right.$$
$$\left\lfloor \frac{(2q+s)(2q+s-1)}{2q} \right\rfloor + (2q+s)(m-2) + \binom{m-2}{2}q - \frac{m-2}{2} \cdot q$$
$$\left. + \frac{m-2}{2}(2q-1) \right\}.$$

The last element of the set is equal to

$$\left\lfloor \frac{(mq+s)(mq+s-1)}{2q} \right\rfloor = \max \ Spec(mq+s).$$

For the situation m is odd and $m > 3$, we have

$$Spec(n) \supseteq \left\{ x : x = y + (3q+s)(m-3) + \binom{m-3}{2}q - \frac{m-3}{2} \cdot q \right.$$
$$+ z_1 + z_2 + \ldots + z_{\frac{m-3}{2}},$$
$$\left. \text{where } y \in Spec(3q+s) \text{ and } z_i \in Spec(2q), i = 1, 2, \ldots , \frac{m-3}{2} \right\}.$$

This concludes this case.

Proposition 2.5. *For* $1 \leq s \leq q$, $m \geq 4$,

$$Spec(mq+s) = \left\{ q+s+\left\lceil \frac{s(q+s)}{2q} \right\rceil + (2q+s)(m-2) + \binom{m-2}{2}q, \ldots , \left\lfloor \binom{mq+s}{2} \Big/ q \right\rfloor \right\}$$
$$= \left\{ \left\lceil \frac{(mq+s)(mq+s-q)}{2q} \right\rceil, \ldots , \left\lfloor \binom{mq+s}{2} \Big/ q \right\rfloor \right\}.$$

Combining the above propositions, we have proved Theorem 1.3. As a special case when $\binom{mq+s}{2}/q$ is an integer, we obtain a star system of order $mq+s$ and also in this case we have the spectrum of partial maximal star systems.

3. Acknowledgement.

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